# Contents

# PART I

# Chapter 1

# Preliminary Results

We take as our starting point the text *Finite Group Theory* [FGT], although we need only a fraction of the material in that text. Frequently quoted results from [FGT] will be recorded in this chapter and in other of the introductory chapters.

Chapters 1 and 2 record some of the most basic terminology and notation we will be using plus some elementary results. The reader should consult [FGT] for other basic group theoretic terminology and notation, although we will try to recall such notation when it is first used, or at least give a specific reference to [FGT] at that point. There is a "List of Symbols" at the end of [FGT] which can be used to help hunt down notation.

We begin in Section 1 with a brief discussion of abstract representations of groups. Then in Section 2 we specialize to permutation representations. In Section 3 we consider graphs and in Section 4 geometries (in the sense of J. Tits) and geometric complexes. In the last few sections of the chapter we record a few basic facts about the general linear group and fiber products of groups.

## 1. Abstract representations

Let $\mathcal{C}$ be a category. For $X$ an object in $\mathcal{C}$, we write $Aut(X)$ for the group of automorphisms of $X$ under the operation of composition in $\mathcal{C}$ (cf. Section 2 in [FGT]). A *representation* of a group $G$ in the category $\mathcal{C}$ is a group homomorphism $\pi; G \to Aut(X)$. For example, a *permutation representation* is a representation in the category of sets and a *linear* *representation* is a representation in the category of vector spaces and linear maps.

If $\alpha : A \to B$ is an isomorphism of objects in $\mathcal{C}$ then $\alpha$ induces a map

$$\alpha^* : Mor(A,A) \to Mor(B,B)$$
$$\beta \mapsto \alpha^{-1}\beta\alpha$$

and $\alpha^*$ restricts to an isomorphism $\alpha^* : Aut(A) \to Aut(B)$. Thus in particular if $A \cong B$ then $Aut(A) \cong Aut(B)$.

A representation $\pi : G \to Aut(A)$ is *faithful* if $\pi$ is injective.

Two representations $\pi : G \to Aut(A)$ and $\sigma : G \to Aut(B)$ in $\mathcal{C}$ are *equivalent* if there exists an isomorphism $\alpha : A \to B$ such that $\sigma = \pi\alpha^*$ is the composition of $\pi$ with $\alpha^*$. Equivalently for all $g \in G$, $(g\pi)\alpha = \alpha(g\sigma)$.

Similarly if $\pi_i : G_i \to Aut(A_i)$, $i = 1,2$, are representations of groups $G_i$ on objects $A_i$ in $\mathcal{C}$, then $\pi_1$ is said to be *quasiequivalent* to $\pi_2$ if there exists a group isomorphism $\beta : G_1 \to G_2$ and an isomorphism $\alpha : A_1 \to A_2$ such that $\pi_2 = \beta^{-1}\pi_1\alpha^*$. Observe that we have a permutation representation of $Aut(G)$ on the equivalence classes of representations of $G$ via $\alpha : \pi \mapsto \alpha\pi$ with the orbits the quasiequivalence classes. Write $Aut(G)_\pi$ for the stabilizer of the equivalence class of $\pi$ under this representation. The following result is Exercise 1.7 in [FGT]:

**Lemma 1.1:** *Let $\pi, \sigma : G \to Aut(A)$ be faithful representations. Then*

*(1)* *$\pi$ is quasiequivalent to $\sigma$ if and only if $G\pi$ is conjugate to $G\sigma$ in $Aut(A)$.*

*(2)* $Aut_{Aut(A)}(G\pi) \cong Aut(G)_\pi$.

If $H \leq G$ then write $Aut_G(H) = N_G(H)/C_G(H)$ for the group of automorphisms of $H$ induced by $G$. Also

$$C_G(H) = \{c \in G : ch = hc \text{ for all } h \in H\}$$

is the *centralizer* in $G$ of $H$ and $N_G(H)$ is the *normalizer* in $G$ of $H$, that is, the largest subgroup of $G$ in which $H$ is normal.

## 2. Permutation representations

In this section $X$ is a set. We refer the reader to Section 5 of [FGT] for our notational conventions involving permutation groups, although we record a few of the most frequently used conventions here. In particular we write $Sym(X)$ for the symmetric group on $X$ and if $X$ is finite we write $Alt(X)$ for the alternating group on $X$. Further $S_n, A_n$ denote the symmetric and alternating groups of degree $n$; that is, $S_n = Sym(X)$ and $A_n = Alt(X)$ for $X$ of order $n$.

Let $\pi : G \to Sym(X)$ be a permutation representation of a group $G$ on $X$. Usually we suppress $\pi$ and write $xg$ for the image $x(g\pi)$ of a point $x \in X$ under the permutation $g\pi$, $g \in G$. For $S \subseteq G$, we write $Fix(S) = Fix_X(S)$ for the set of fixed points of $S$ on $X$. For $Y \subseteq X$,

$$G_Y = \{g \in G : yg = y \text{ for all } y \in Y\}$$

is the *pointwise stabilizer* of $Y$ in $G$,

$$G(Y) = \{g \in G : Yg = Y\}$$

is the *global stabilizer* of $Y$ in $G$, and $G^Y = G(Y)/G_Y$ is the image of $G(Y)$ in $Sym(Y)$ under the restriction map. In particular $G_y$ denotes the stabilizer of a point $y \in X$.

Recall the *orbit* of $x \in X$ under $G$ is $xG = \{xg : g \in G\}$ and $G$ is *transitive* on $X$ if $G$ has just one orbit on $X$. If $G$ is transitive on $X$ then our representation $\pi$ is equivalent to the representation of $G$ by right multiplication on the coset space $G/G_x$ via the map $G_x g \mapsto xg$ (cf. 5.9 in [FGT]).

A subgroup $K$ of $G$ is a *regular normal subgroup* of $G$ if $K \trianglelefteq G$ and $K$ is *regular* on $X$; that is, $K$ is transitive on $X$ and $K_x = 1$ for $x \in X$.

Recall a transitive permutation group $G$ is *primitive* on $X$ if $G$ preserves no nontrivial partition on $X$. Further $G$ is primitive on $X$ if and only if $G_x$ is maximal in $G$ (cf. 5.19 in [FGT]).

**Lemma 2.1:** *Let $G$ be transitive on $X$, $x \in X$, and $K \leq G$. Then*

*(1) $K$ is transitive on $X$ if and only if $G = G_x K$.*
*(2) If $1 \neq K \trianglelefteq G$ and $G$ is primitive on $X$ then $K$ is transitive on $X$.*
*(3) If $K$ is a regular normal subgroup of $G$ then the representations of $G_x$ on $X$ and on $K$ by conjugation are equivalent.*

**Proof:** These are all well known; see, for example, 5.20, 15.15, and 15.11 in [FGT].

Recall that $G$ is *$t$-transitive* on $X$ if $G$ is transitive on ordered $t$-tuples of distinct points of $X$. In Chapter 6 we will find that the Mathieu group $M_{m+t}$ is $t$-transitive on $m+t$ points for $m = 19$ and $t = 3, 4, 5$ and $m = 7$ and $t = 4, 5$.

**Lemma 2.2:** *Let $G$ be $t$-transitive on a finite set $X$ with $t \geq 2$, $x \in X$, and $1 \neq K \trianglelefteq G$. Then*

*(1) $G$ is primitive on $X$.*
*(2) $K$ is transitive on $X$ and $G = G_x K$.*

*(3) If $K$ is regular on $X$ then $|K| = |X| = p^e$ is a power of some prime $p$, and if $t > 2$ then $p = 2$.*

*(4) If $t = 3 < |X|$ and $|G : K| = 2$ then $K$ is 2-transitive on $X$.*

**Proof:** Again these are well-known facts. See, for example, 15.14 and 15.13 in [FGT] for (1) and (3), respectively. Part (2) follows from (1) and 1.1. Part (4) is left as Exercise 1.1.

## 3. Graphs

A *graph* $\Delta = (\Delta, *)$ consists of a set $\Delta$ of *vertices* (or objects or points) together with a symmetric relation $*$ called *adjacency* (or incidence or something else). The ordered pairs in the relation are called the *edges* of the graph. We write $u * v$ to indicate two vertices are related via $*$ and say $u$ is *adjacent* to $v$. Denote by $\Delta(u)$ the set of vertices adjacent to $u$ and distinct from $u$ and define $u^\perp = \Delta(u) \cup \{u\}$.

A *path of length $n$* from $u$ to $v$ is a sequence of vertices $u = u_0, u_1, \ldots, u_n = v$ such that $u_{i+1} \in u_i^\perp$ for each $i$. Denote by $d(u, v)$ the minimal length of a path from $u$ to $v$. If no such path exists set $d(u, v) = \infty$. $d(u, v)$ is the *distance* from $u$ to $v$.

The relation $\sim$ on $\Delta$ defined by $u \sim v$ if and only if $d(u, v) < \infty$ is an equivalence relation on $\Delta$. The equivalence classes of this relation are called the *connected components* of the graph. The graph is *connected* if it has just one connected component. Equivalently there is a path between any pair of vertices.

A *morphism* of graphs is a function $\alpha : \Delta \to \Delta'$ from the vertex set of $\Delta$ to the vertex set of $\Delta'$ which preserves adjacency; that is, $u^\perp\alpha \subseteq (u\alpha)^\perp$ for each $u \in \Delta$.

A group $G$ of automorphisms of $\Delta$ is *edge transitive* on $\Delta$ if G is transitive on $\Delta$ and on the edges of $\Delta$.

Representations of groups on graphs play a big role in this book. For example, we prove the uniqueness of some of the sporadics $G$ by considering a representation of $G$ on a suitable graph. The following construction supplies us with such graphs.

Let $G$ be a transitive permutation group on a finite set $\Delta$. Recall the *orbitals* of $G$ on $\Delta$ are the orbits of $G$ on the set product $\Delta^2 = \Delta \times \Delta$. The *permutation rank* of $G$ is the number of orbitals of $G$; recall this is also the number of orbits of $G_x$ on $\Delta$ for $x \in \Delta$.

Given an orbital $\Omega$ of G, the *paired orbital* $\Omega^p$ of $\Omega$ is

$$\Omega^p = \{(y, x) : (x, y) \in \Omega\}.$$

Evidently $\Omega^p$ is an orbital of $G$ with $(\Omega^p)^p = \Omega$. The orbital $\Omega$ is said to be *self-paired* if $\Omega^p = \Omega$. For example, the *diagonal orbital* $\{(x, x) : x \in \Delta\}$ is a self-paired orbital.

**Lemma 3.1:** *(1) A nondiagonal orbital $(x, y)G$ of $G$ is self-paired if and only if $(x, y)$ is a cycle in some $g \in G$.*

*(2) If $G$ is finite then $G$ possesses a nondiagonal self-paired orbital if and only if $G$ is of even order.*

*(3) If $G$ is of even order and permutation rank 3 then all orbitals of $G$ are self-paired.*

**Proof:** See 16.1 in [FGT].

**Lemma 3.2:** *(1) Let $\Omega$ be a self-paired orbital of $G$. Then $\Omega$ is a symmetric relation on $\Delta$, so $\Delta = (\Delta, \Omega)$ is a graph and $G$ is an edge transitive group of automorphisms of $\Delta$.*

*(2) Conversely if $H$ is an edge transitive group of automorphisms of a graph $\Delta = (\Delta, *)$ then the set $*$ of edges of $\Delta$ is a self-paired orbital of $G$ on $\Delta$, and $\Delta$ is the graph determined by this orbital.*

Many of the sporadics have representations as rank 3 permutation groups. Indeed some were discovered via such representations; see Chapter 5 for a discussion of the sporadics discovered this way. See also Exercise 16.5, which considers the rank 3 representation of $J_2$, and Lemmas 24.6, 24.7, and 24.11, which establish the existence of rank 3 representations of $Mc$, $U_4(3)$, and $HS$.

In the remainder of this section assume $G$ is of even order and permutation rank 3 on a set $X$. Hence $G$ has two nondiagonal orbitals $\Delta$ and $\Gamma$ and by 3.1, each is self-paired. Further for $x \in X$, $G_x$ has two orbits $\Delta(x)$ and $\Gamma(x)$ on $X - \{x\}$, where $\Delta(x) = \{y \in X : (x, y) \in \Delta\}$ and $\Gamma(x) = \{z \in X : (x, z) \in \Gamma\}$. By 3.2, $X = (X, \Delta)$ is a graph and $G$ is an edge transitive group of automorphisms of $X$. Notice $\Delta(x) = X(x)$ in our old notation.

The following notation is standard for rank 3 groups and their graphs: $k = |\Delta(x)|$, $l = |\Gamma(x)|$, $\lambda = |\Delta(x) \cap \Delta(y)|$ for $y \in \Delta(x)$, and $\mu = |\Delta(x) \cap \Delta(z)|$ for $z \in \Gamma(x)$. The integers $k, l, \lambda, \mu$ are the *parameters* of the rank 3 group $G$. Also let $n = |X|$ be the degree of the representation.

**Lemma 3.3:** *Let $G$ be a rank 3 permutation group of even order on a finite set of order $n$ with parameters $k, l, \lambda, \mu$. Then*

*(1)* $n = k + l + 1$.

*(2)* $\mu l = k(k - \lambda - 1)$.

(3) *If $\mu \neq 0$ or $k$ then $G$ is primitive and the graph $\mathcal{G}$ of $G$ is connected.*

(4) *Assume $G$ is primitive. Then either*

  (a) *$k = l$ and $\mu = \lambda + 1 = k/2$, or*

  (b) *$d = (\lambda - \mu)^2 + 4(k - \mu)$ is a square and setting $D = 2k + (\lambda - \mu)(k + l)$, $d^{1/2}$ divides $D$ and $2d^{1/2}$ divides $D$ if and only if $n$ is odd.*

**Proof:** See Section 16 of [FGT].

## 4. Geometries and complexes

In this book we adopt a notion of geometry due to J. Tits in [T1].

Let $I$ be a finite set. For $J \subseteq I$, let $J' = I - J$ be the complement of $J$ in $I$. A *geometry* over $I$ is a triple $(\Gamma, \tau, *)$ where $\Gamma$ is a set of objects, $\tau : \Gamma \to I$ is a surjective type function, and $*$ is a symmetric incidence relation on $\Gamma$ such that objects $u$ and $v$ of the same type are incident if and only if $u = v$. We call $\tau(u)$ the *type* of the object $u$. Notice $(\Gamma, *)$ is a graph. We usually write $\Gamma$ for the geometry $(\Gamma, \tau, *)$ and $\Gamma_i$ for the set of objects of $\Gamma$ of type $i$.

The *rank* of the geometry $\Gamma$ is the cardinality of $I$.

A *flag* of $\Gamma$ is a subset $T$ of $\Gamma$ such that each pair of objects in $T$ is incident. Notice our one axiom insures that if $T$ is a flag then the type function $\tau : T \to I$ is injective. Define the *type* of $T$ to be $\tau(T)$ and the *rank* of $T$ to be the cardinality of $T$. The *chambers* of $\Gamma$ are the flags of type $I$.

A *morphism* $\alpha : \Gamma \to \Gamma'$ of geometries is a function $\alpha : \Gamma \to \Gamma'$ of the associated object sets which preserves type and incidence; that is, if $u, v \in \Gamma$ with $u * v$ then $\tau(u) = \tau'(u\alpha)$ and $u\alpha *' v\alpha$. A group $G$ of automorphisms of $\Gamma$ is *edge transitive* if $G$ is transitive on flags of type $J$ for each subset $J$ of $I$ of order at most 2. Similarly $G$ is *flag transitive* on $\Gamma$ if $G$ is transitive on flags of type $J$ for all $J \subseteq I$.

Representations of groups on geometries also play an important role in *Sporadic Groups*. For example, the Steiner systems in Chapter 6 are rank 2 geometries whose automorphism groups are the Mathieu groups. Here are some other examples:

**Examples** (1) Let $V$ be an $n$-dimensional vector space over a field $F$. We associate a geometry $PG(V)$ to $V$ called the *projective geometry* of $V$. The objects of $PG(V)$ are the proper nonzero subspaces of $V$, with incidence defined by inclusion. The type of $U$ is $\tau(U) = dim(U)$. Thus

$PG(V)$ is of rank $n-1$. The projective general linear group on $V$ is a flag transitive group of automorphism of $PG(V)$.

(2) A *projective plane* is a rank 2 geometry $\Gamma$ whose two types of objects are called points and lines and such that:

(PP1) Each pair of distinct points is incident with a unique line.
(PP2) Each pair of distinct lines is incident with a unique point.
(PP3) There exist four points no three of which are on a common line.

**Remarks.** (1) Rank 2 projective geometries are projective planes.

(2) If $\Gamma$ is a finite projective plane then there exists an integer $q$ such that each point is incident with exactly $q+1$ lines, each line is incident with exactly $q+1$ points, and $\Gamma$ has $q^2+q+1$ points and lines.

**Examples** (3) If $f$ is a sesquilinear or quadratic form on $V$ then the *totally singular subspaces* of $V$ are the subspaces $U$ such that $f$ is trivial on $U$. The set of such subspaces forms a subgeometry of the projective geometry. See, for example, page 99 in [FGT].

(4) Let $G$ be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of $G$. Define $\Gamma(G,\mathcal{F})$ to be the geometry whose set of objects of type $i$ is the coset space $G/G_i$ and with objects $G_i x$ and $G_j y$ incident if $G_i x \cap G_j y \neq \varnothing$. Observe:

**Lemma 4.1:** *(1) $G$ is represented as an edge transitive group of automorphisms of $\Gamma(G,\mathcal{F})$ via right multiplication and $\Gamma(G,\mathcal{F})$ possesses a chamber.*

*(2) Conversely if $H$ is an edge transitive group of automorphisms of a geometry $\Gamma$ and $\Gamma$ possesses a chamber $C$, then $\Gamma \cong \Gamma(H,\mathcal{F})$, where $\mathcal{F} = (H_c : c \in C)$.*

The construction of 4.1 allows us to represent each group $G$ on various geometries. The construction is used in Chapter 13 as part of our machine for establishing the uniqueness of groups. Further the construction associates to each sporadic group $G$ various geometries which can be used to study the subgroup structure of $G$. The latter point of view is not explored to any extent in *Sporadic Groups*; see instead [A2] or [RS] where such geometries are discussed. We do use the 2-local geometry of $M_{24}$ to study that group in Chapter 7.

Define the *direct sum* of geometries $\Gamma_i$ on $I_i$, $i = 1,2$, to be the geometry $\Gamma_1 \oplus \Gamma_2$ over the disjoint union $I$ of $I_1$ and $I_2$ whose object set is the disjoint union of $\Gamma_1$ and $\Gamma_2$, whose type function is $\tau_1 \cup \tau_2$, and whose incidence is inherited from $\Gamma_1$ and $\Gamma_2$ with each object in $\Gamma_1$ incident with each object in $\Gamma_2$.

**Example** (5) A *generalized digon* is a rank 2 geometry which is the direct sum of rank 1 geometries. That is, each element of type 1 is incident with each element of type 2.

**Lemma 4.2:** *Let $G$ be a group and $\mathcal{F} = \{G_1, G_2\}$ a pair of subgroups of $G$. Then $\Gamma(G, \mathcal{F})$ is a generalized digon if and only if $G = G_1G_2$.*

**Proof:** As $G$ is edge transitive on $\Gamma$, $\Gamma$ is a generalized digon if and only if $G_2$ is transitive on $\Gamma_1$ if and only if $G = G_1G_2$ by 2.1.1.

Given a flag $T$, let $\Gamma(T)$ consist of all $v \in \Gamma - T$ such that $v * t$ for all $t \in T$. We regard $\Gamma(T)$ as a geometry over $I - \tau(T)$. The geometry $\Gamma(T)$ is called the *residue* of $T$.

**Example** (6) Let $\Gamma = PG(V)$ be the projective geometry of an $n$-dimensional vector space. Then for $U \in \Gamma$, the residue $\Gamma(U)$ of the object $U$ is isomorphic to $PG(U) \oplus PG(V/U)$.

The category of geometries is not large enough; we must also consider either the category of chamber systems or the category of geometric complexes.

A *chamber system* over $I$ is a set $X$ together with a collection of equivalence relations $\sim_i$, $i \in I$. For $J \subseteq I$ and $x \in X$, let $\sim_J$ be the equivalence relation generated by the relations $\sim_j$, $j \in J$, and $[x]_J$ the equivalence class of $\sim_J$ containing $x$. Define $X$ to be *nondegenerate* if for each $x \in X$, and $j \in I$, $\{x\} = \bigcap_i [x]_{i'}$ and $[x]_j = \bigcap_{i \in j'} [x]_{i'}$. A morphism of chamber systems over $I$ is a map preserving each equivalence relation.

The notion of "chamber system" was introduced by J. Tits in [T1].

Recall that a *simplicial complex* $K$ consists of a set $X$ of *vertices* together with a distinguished set of nonempty subsets of $X$ called the *simplices* of $K$ such that each nonempty subset of simplex is a simplex. The morphisms of simplicial complexes are the *simplicial maps*; that is, a simplicial map $f : K \to K'$ is a map $f : X \to X'$ of vertices such that $f(s)$ is a simplex of $K'$ for each simplex $s$ of $K$.

**Example** (7) If $\Delta$ is a graph then the *clique complex* $K(\Delta)$ is the simplicial complex whose vertices are the vertices of $\Delta$ and whose simplices are the finite cliques of $\Delta$. Recall a *clique* of $\Delta$ is a set $Y$ of vertices such that $y \in x^{\perp}$ for each $x, y \in Y$. Conversely if $K$ is a simplicial complex then the *graph* of $K$ is the graph $\Delta = \Delta(K)$ whose vertices are the vertices of $K$ and with $x * y$ if $\{x, y\}$ is a simplex of $K$. Observe $K$ is a subcomplex of $K(\Delta(K))$.

Given a simplicial complex $K$ and a simplex $s$ of $K$, define the *star* of $s$ to be the subcomplex $st_K(s)$ consisting of the simplices $t$ of $K$ such that

$s \cup t$ is a simplex of $K$. Define the *link* $Link_K(s)$ to be the subcomplex of $st_K(s)$ consisting of the simplices $t$ of $st_K(s)$ such that $t \cap s = \varnothing$.

A *geometric complex* over $I$ is a geometry $\Gamma$ over $I$ together with a collection $\mathcal{C}$ of distinguished chambers of $\Gamma$ such that each flag of rank 1 or 2 is contained in a member of $\mathcal{C}$. The *simplices* of the complex are the subflags of members of $\mathcal{C}$. A morphism $\alpha : \mathcal{C} \to \mathcal{C}'$ of complexes over $I$ is a morphism of geometries with $\mathcal{C}\alpha \subseteq \mathcal{C}'$. Notice a geometric complex is just a simplicial complex together with a type function on vertices that is injective on simplices.

**Example** (8) The *flag complex* of a geometry $\Gamma$ is the simplicial complex on $\Gamma$ in which all chambers are distinguished. Notice the flag complex is a geometric complex if and only if each flag of rank at most 2 is contained in a chamber. Further as a simplicial complex, the flag complex is just the clique complex of $\Gamma$ regarded as a graph.

Many theorems about geometries are best established in the larger categories of geometric complexes or chamber systems. Theorem 4.11 is an example of such a result. We find in a moment in Lemma 4.3 below that the category of nondegenerate chamber systems is isomorphic to the category of geometric complexes. I find the latter category more intuitive and so work with complexes rather than chamber systems. But others prefer chamber systems and there is a growing literature on the subject.

Given a chamber system $X$ define $\Gamma_X$ to be the geometry whose objects of type $i$ are the equivalence classes of the relation $\sim_{i'}$ with $A * B$ if and only if $A \cap B \neq \varnothing$. For $x \in X$ let $C_x$ be the set of equivalence classes containing $x$; thus $C_x$ is a chamber in $\Gamma_X$. Define $\mathcal{C}_X$ to be the set of chambers $C_x$, $x \in X$, of $\Gamma_X$. If $\alpha : X \to X'$ is a morphism of chamber systems define $\alpha_{\mathcal{C}} : \mathcal{C}_X \to \mathcal{C}_{X'}$ to be the morphism of complexes such that $\alpha_{\mathcal{C}} : A \mapsto A'$ for $A$ a $\sim_{i'}$ equivalence class of $X$ and $A'$ the $\sim_{i'}$ equivalence class containing $A\alpha$.

Conversely given a geometric complex $\mathcal{C}$ over $I$ let $\sim_i$ be the equivalence relation on $\mathcal{C}$ defined by $A \sim_i B$ if $A$ and $B$ have the same subflag of type $i'$. Then we have a chamber system $X_{\mathcal{C}}$ with chamber set $\mathcal{C}$ and equivalence relations $\sim_i$. Further if $\alpha : \mathcal{C} \to \mathcal{C}'$ is a morphism of complexes let $\alpha_X : X_{\mathcal{C}} \to X_{\mathcal{C}'}$ be the morphism of chamber systems defined by the induced map on chambers.

**Lemma 4.3:** *The category of nondegenerate chamber systems over $I$ is isomorphic to the category of geometric complexes over $I$ via the maps $X \mapsto \mathcal{C}_X$ and $\mathcal{C} \mapsto X_{\mathcal{C}}$.*

**Example** (9) Let $G$ be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of $I$. For $J \subseteq I$ and $x \in G$ define

$$S_{J,x} = \{G_j x : j \in J\}.$$

Thus $S_{J,x}$ is a flag of the geometry $\Gamma(G,\mathcal{F})$ of type $J$. Observe that the stabilizer of the flag $S_J = S_{J,1}$ is the subgroup $G_J = \bigcap_{j\in J} G_j$. Define $\mathcal{C}(G,\mathcal{F})$ to be the geometric complex over $I$ with geometry $\Gamma(G,\mathcal{F})$ and distinguished chambers $S_{I,x}$, $x \in G$. Then $\mathcal{C}(G,\mathcal{F})$ is a geometric complex with simplices $S_{J,x}$, $J \subseteq I$, $x \in G$, and $G$ acts as an edge transitive group of automorphisms of $\mathcal{C}(G,\mathcal{F})$ via right multiplication, and transitively on $\mathcal{C}(G,\mathcal{F})$. Indeed:

**Lemma 4.4:** *Assume $\mathcal{C}$ is a geometric complex over $I$ and $G$ is an edge transitive group of automorphisms with $\mathcal{C} = CG$ for some $C \in \mathcal{C}$. Let $G_i = G_{x_i}$, where $x_i \in C$ is of type $i$, and let $\mathcal{F} = (G_i : i \in I)$. Then the map $x_i g \mapsto G_i g$ is an isomorphism of $\mathcal{C}$ with $\mathcal{C}(G,\mathcal{F})$.*

Further we have a chamber system $X(G,\mathcal{F})$ whose chamber set is $G/G_I$ and with $G_I x \sim_i G_I y$ if and only if $xy^{-1} \in G_{i'}$. Observe that the map $G_I x \mapsto S_{I,x}$ defines an isomorphism of the chamber systems $X(G,\mathcal{F})$ and $X_{\mathcal{C}(G,\mathcal{F})}$.

The construction of 4.4 allows us to represent a group $G$ on many complexes. We make use of this construction in Chapter 13 as part of our uniqueness machine.

Let $\mathcal{C} = (\Gamma, \mathcal{C})$ be a geometric complex over $I$. Given a simplex $S$ of type $J$, regard the link $Link_{\mathcal{C}}(S)$ of $S$ to be a geometric complex over $J'$; thus the objects of $Link_{\mathcal{C}}(S)$ of type $i \in J'$ are those $v \in \Gamma_i$ such that $S \cup \{v\}$ is a simplex and with $v * u$ if $S \cup \{u,v\}$ is a simplex, and the chamber set $\mathcal{C}(S)$ of $Link_{\mathcal{C}}(S)$ consists of the simplices $C - S$ with $S \subseteq C \in \mathcal{C}$. For example, $\mathcal{C} = Link_{\mathcal{C}}(\varnothing)$ is the link of the empty simplex. Notice that if all flags are simplices then the geometry of $Link_{\mathcal{C}}(S)$ is the residue $\Gamma(S)$ of $S$ in the geometry $\Gamma$.

We say $\mathcal{C}$ is *residually connected* if the link of each simplex of corank at least two (including $\varnothing$ if $|I| \geq 2$) is connected. A geometry $\Gamma$ is residually connected if each flag is contained in a chamber and the flag complex of $\Gamma$ is residually connected.

**Lemma 4.5:** *Let $\mathcal{F} = (G_i : i \in I)$ be a family of subgroups of $G$. Then*

*(1)* $\Gamma(G,\mathcal{F})$ *is connected if and only if* $G = \langle \mathcal{F} \rangle$.

*(2)* $Link_{\mathcal{C}}(S_J) \cong \mathcal{C}(G_J, \mathcal{F}_J)$ *for each* $J \subseteq I$, *where*

$$\mathcal{F}_J = (G_{J\cup\{i\}} : i \in J').$$

(3) *$\mathcal{C}(G,\mathcal{F})$ is residually connected if and only if $G_J = \langle \mathcal{F}_J \rangle$ for all $J \subseteq I$.*

**Proof:** Notice (1) and (2) imply (3) so it remains to prove (1) and (2).

As $\mathcal{F}$ is a chamber, the connected component $\Delta$ of $G_i$ in $\Gamma$ is the same for each $i$, and $H = \langle \mathcal{F} \rangle$ acts on $\Delta$. Conversely as $G_i$ is transitive on $\Gamma_j(G_i)$ for each $j$, $\Delta \subseteq \Delta' = \bigcup_j G_j H$, so $\Delta = \Delta'$ and $H$ is transitive on $\Gamma_i \cap \Delta$ for each $i$. Thus as $G$ is transitive on $\Gamma_i$, $\Gamma$ is connected if and only if $H$ is transitive on $\Gamma_i$ for each $i$, and as $G_i \leq H$ this holds if and only if $G = H$. Thus (1) is established.

In (2) the desired isomorphism is $G_k x \mapsto S_{K,x}$ for $x \in G_J$, $K = J \cup \{k\}$.

**Lemma 4.6:** *Assume $\mathcal{C}$ is a residually connected geometric complex over $I$, $J \subseteq I$ with $|J| \geq 2$, and $x, y \in \Gamma$. Then there exists a path $x = v_0, \ldots, v_m = y$ in $\Gamma$ with $\tau(v_i) \in J$ for all $0 < i < m$.*

**Proof:** Choose $x, y$ to be a counterexample with $d = d(x,y)$ minimal. As the residue $\Gamma$ of the simplex $\varnothing$ is connected, $d$ is finite, and clearly $d > 1$. Let $x = v_0 \cdots v_d = y$ be a path. By minimality of $d$ there is a path $v_1 = u_0 \cdots u_m = y$ with $\tau(u_i) \in J$ for $0 < i < m$. Thus if $\tau(v_1) \in J$ then $xu_0 \cdots u_m$ is the desired path, so assume $\tau(v_1) \notin J$.

We also induct on the rank of $\mathcal{C}$; if the rank is 2 the lemma is trivial, so our induction is anchored. Now $Link_{\mathcal{C}}(v_1)$ is a residually connected complex and $x, u_1 \in Link_{\mathcal{C}}(v_1)$, so by induction on the rank of $\mathcal{C}$, there is a path $x = w_0 \cdots w_k = u_1$ with $\tau(w_i) \in J$ for $0 < i < k$. Now $x = w_0 \cdots w_k u_2 \cdots u_m = y$ does the job.

Given geometric complexes $\mathcal{C}$ over $J$ and $\bar{\mathcal{C}}$ over $\bar{J}$ define $\mathcal{C} \oplus \bar{\mathcal{C}}$ to be the geometric complex over the disjoint union $I$ of $J$ and $\bar{J}$ whose geometry is $\Gamma \oplus \bar{\Gamma}$ and with chamber set $\{C \cup \bar{C} : C \in \mathcal{C}, \bar{C} \in \bar{\mathcal{C}}\}$.

The *basic diagram* for a geometric complex $\mathcal{C}$ over $I$ is the graph on $I$ obtained by joining distinct $i, j$ in $I$ if for some simplex $T$ of type $\{i,j\}'$ (including $\varnothing$ if $|I| = 2$), $Link_{\mathcal{C}}(T)$ is *not* a generalized digon. The basic diagram of a geometry is the basic diagram of its flag complex.

Diagrams containing more information can also be associated to each geometry or geometric complex. The study of such diagrams was initiated by Tits [T1] and Buekenout [Bu].

A graph on $I$ is a *string* if we can order $I = \{1, \ldots, n\}$ so that the edges of $I$ are $\{i, i+1\}$, $1 \leq i < n$. Such an ordering will be termed a *string ordering*. A *string geometry* is a geometry whose basic diagram is a string. Most of the geometries considered in *Sporadic Groups* are string geometries; for example:

**Example** (10) The basic diagram of projective geometry is a string.

**Lemma 4.7:** *Assume $C$ is a residually connected geometric complex such that $I = I_1 + I_2$ is a partition of $I$ such that $I_1$ and $I_2$ are unions of connected components of the basic diagram of $I$. Then $C = C^1 \oplus C^2$, where $C^i$ consists of the simplices of type $I_i$.*

**Proof:** We may assume $I_i \neq \varnothing$ for $i = 1, 2$. By definition of the basic diagram, the lemma holds if $\Gamma$ is of rank 2. Thus we may assume $I_1$ has rank at least 2. Let $x_i \in \Gamma^i$; by 4.6 there exists a path $x_1 = v_0 \cdots v_m = x_2$ with $\tau(v_i) \in I_1$ for $i < m$. Choose this path with $m$ minimal; if $m = 1$ for each choice of $x_i$ we are done, so choose $x_i$ such that $m$ is minimal subject to $m > 1$. Then of course $m = 2$, so $x_i \in Link_C(v_1)$. But by induction on the rank of $\Gamma$, $x_1$ is incident with $x_2$ in $Link_C(v_1)$, and hence also in $\Gamma$.

The proof of the following result is trivial:

**Lemma 4.8:** *If $C$ is a geometric complex then the following are equivalent:*

*(1)* *All flags of $\Gamma$ are simplices.*
*(2)* *$Link_C(S) = \Gamma(S)$ for each simplex $S$ of $C$.*

**Lemma 4.9:** *Assume $C$ is a residually connected geometric complex such that the connected components of the basic diagram of $C$ are strings. Then all flags of $C$ are simplices.*

**Proof:** Assume not and let $T$ be a flag of minimal rank $m$ which is not a simplex. As $C$ is a geometric complex, $m > 2$. Pick a string ordering for $I$ and let $T = \{x_1, \dots, x_m\}$ with $\tau(x_i) < \tau(x_{i+1})$. Let $x = x_2$. By minimality of $m$, $\{x_1, x\}$ and $\{x_2, \dots, x_m\}$ are simplices. Further by 4.7, $Link_C(x) = C^1 \oplus C^2$, where $C^i$ is the subgeometry on $I_i$, $I_1 = \{1\}$, and $I_2 = \{3, \dots, n\}$. Thus $\{x_1, x_3, \dots, x_m\}$ is a simplex in $Link_C(x)$, so $T$ is a simplex of $C$.

**Lemma 4.10:** *Let $G$ be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of $G$, and assume $C = C(G, \mathcal{F})$ is residually connected. Then the following are equivalent:*

*(1)* *$G$ is flag transitive on $\Gamma(G, \mathcal{F})$.*
*(2)* *Each flag of $\Gamma(G, \mathcal{F})$ is a simplex.*
*(3)* *$\Gamma(S_J) = Link_C(S_J) \cong \Gamma(G_J, \mathcal{F}_J)$ for each $J \subseteq I$.*

**Proof:** By 4.5.2 and 4.8, (2) and (3) are equivalent. As $G$ is transitive on simplices of $C$ of type $J$ for each $J \subseteq I$, (1) and (2) are equivalent.

**Theorem 4.11:** *Let $G$ be a group, $I = \{1, \dots, n\}$, and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of $G$. Assume*

*(a) $\mathcal{C}(G, \mathcal{F})$ is residually connected; that is, $G_J = \langle \mathcal{F}_J \rangle$ for all $J \subseteq I$.*
*(b) The diagram of $\mathcal{C}(G, \mathcal{F})$ is a union of strings; that is, $\langle G_{i'}, G_{j'} \rangle = G_{i'} G_{j'}$ for all $i, j \in I$ with $|i - j| > 1$.*

*Then*

*(1) $G$ is flag transitive on $\Gamma(G, \mathcal{F})$.*
*(2) $\Gamma(S_J) \cong \Gamma(G_J, \mathcal{F}_J)$ for all $J \subseteq I$.*

**Proof:** This follows from 4.9 and 4.10. Use 4.5 to see that the conditions of (a) are equivalent and 4.2 to see that the conditions of (b) are equivalent.

## 5. The general linear group and its projective geometry

In this section $F$ is a field, $n$ is a positive integer, and $V$ is an $n$-dimensional vector space over $F$. Recall that the group of vector space automorphisms of $V$ is the *general linear group* $GL(V)$. We assume the reader is familiar with basic facts about $GL(V)$, such as can be found in Section 13 of [FGT]. For example, as the isomorphism type of $V$ depends only on $n$ and $F$, the same is true for $GL(V)$, so we can also write $GL_n(F)$ for $GL(V)$.

Recall that from Section 13 in [FGT] that each ordered basis $X = (x_1, \dots, x_n)$ of $V$ determines an isomorphism $M_X$ of $GL(V)$ with the group of all nonsingular $n$-by-$n$ matrices over $F$ defined by $M_X(g) = (g_{ij})$, where for $g \in GL(V)$, $g_{ij} \in F$ is defined by $x_i g = \sum_j g_{ij} x_j$. Thus we will sometimes view $GL(V)$ as this matrix group.

We write $SL(V)$ or $SL_n(F)$ for the subgroup of matrices in $GL(V)$ of determinant 1. Thus $SL_n(F)$ is the *special linear group*. As the kernel of the determinant map, $SL_n(F)$ is a normal subgroup of $GL_n(F)$.

A *semilinear transformation of* $V$ is a bijection $g : V \to V$ that preserves addition and such that there exists $\sigma(g) \in Aut(F)$ such that for each $a \in F$ and $v \in V$, $(av)g = a\sigma(g)v$. Define $\Gamma = \Gamma(V)$ to be the set of all semilinear transformations of $V$. Notice the map $\sigma : \Gamma \to Aut(F)$ is a surjective group homomorphism with kernel $GL(V)$ and $\Gamma(V)$ is the split extension of $GL(V)$ by the group $\{f_\alpha : \alpha \in Aut(F)\} \cong Aut(F)$ of *field automorphisms* determined by the basis $X$ of $V$, where

$$f_\alpha : \sum_i a_i x_i \mapsto \sum_i (a_i \alpha) x_i.$$

Notice also that $\Gamma(V)$ permutes the points of the projective geometry $PG(V)$ and this action induces a representation of $\Gamma(V)$ as a group of automorphisms of $PG(V)$ with kernel the scalar matrices. Thus the image $P\Gamma(V)$ is a group of automorphisms of $PG(V)$ which is the split extension of $PGL(V)$ by the group of field automorphisms.

If $F = GF(q)$ is the finite field of order $q$ we write $GL_n(q)$ for $GL_n(F)$, $SL_n(q)$ for $SL_n(F)$, $PGL_n(q)$ for $PGL_n(F)$, and $L_n(q) = PSL_n(q)$ for $PSL_n(F)$.

See Section 13 in [FGT] for the definition of the *transvections* in $GL(V)$ and properties of transvections.

**Lemma 5.1:** *Let $G = PGL(V)$, $S = PSL(V)$, and $H$ the stabilizer in $G$ of a point $p$ of $PG(V)$. Assume $n \geq 2$. Then*

*(1) $H$ is the split extension of the group $Q$ of all transvections of $V$ with center $p$ by the stabilizer $L$ of $p$ and a hyperplane $U$ of $V$ complementing $p$.*

*(2) $Q \cong U$, $L \cong GL(U)$, and the action of $L$ by conjugation on $Q$ is equivalent to the action of $L$ on $U$.*

*(3) $Q$ is the unique minimal normal subgroup of $H \cap L$.*

**Proof:** Let $\hat{G} = GL(V)$ and regard $\hat{G}$ as a group of matrices relative to a basis $X$ for $V$ such that $p = \langle x_1 \rangle$. Then the preimage $\hat{H}$ of $H$ in $\hat{G}$ consists of all matrices

$$g = \begin{pmatrix} a(g) & 0 \\ \alpha(g) & A(g) \end{pmatrix}$$

with $a(g) \in F^{\#}$, $\alpha(g)$ a row matrix, and $A(g) \in GL(U)$. Moreover $Q$ consists of the matrices $g$ with $a(g) = 1$ and $A(g) = I$, while $\hat{L}$ consists of all matrices $h$ with $\alpha(h) = 0$. Further $g^h \in Q$ with $\alpha(g^h) = a(h)A(h)^{-1}\alpha(g)$. In particular $\hat{H}$ is the split extension of $Q$ by $\hat{L}$, and $Q \cong U$ is abelian. Further $\hat{L} = L_0 \times K$, where $K$ is the group of scalar matrices and $L_0$ consists of those $h \in \hat{L}$ with $a(h) = 1$. Thus the image $L$ of $\hat{L}$ in $G$ is isomorphic to $L_0 \cong GL(U)$, and the action of $L$ by conjugation on $H$ is equivalent to the action of $L \cong L_0$ on $U \cong Q$.

So (1) and (2) are established. Finally as the action of $L$ on $Q$ is equivalent to its action on $U$, $L$ (and even $L \cap S$) is faithful and irreducible on $Q$, so $Q$ is minimal normal in $H$. Now if $M$ is a second minimal normal subgroup of $H$, then $\langle M, Q \rangle = M \times Q$, so $M \leq C_H(Q)$ and $M \cap Q = 1$. But as $H = LQ$ with $L$ faithful on $Q$, $Q = C_H(Q)$, contradicting $M \cap Q = 1$.

The projective plane over the field of order 4 will be the starting point