PHOEBUS J. DHRYMES

Mathematics for Econometrics

THIRD EDITION

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Third Edition

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Preface to the Third Edition

The third edition differs from the second edition in several respects. The coverage of matrix algebra has been expanded. For example, the topic of inverting partitioned matrices in this edition deals explicitly with a problem that arises in estimation under (linear) constraints. Often this problem forces us to deal with a block partitioned matrix whose (1,1) and (2,2) blocks are singular matrices. The standard method for inverting such matrices fails; unless the problem is resolved, explicit representation of estimators and associated Lagrange multipliers is not available. An important application is in estimating the parameters of the general linear structural econometric model, when the identifying restrictions are imposed by means of Lagrange multipliers. This formulation permits a near effortless test of the validity of such (overidentifying) restrictions.

This edition also contains a treatment of the vector representation of restricted matrices such as symmetric, triangular, diagonal and the like. The representation is in terms of restricted linear subspaces. Another new feature is the treatment of permutation matrices and the vec operator, leading to an explicit representation of the relationship between $A \otimes B$ and $B \otimes A$.

In addition, it contains three new chapters, one on asymptotic expansions and two on applications of the material covered in this volume to the general linear model and the general linear structural econometric model, respectively. The salient features of the estimation problems in these two topics are discussed rigorously and succinctly.

This version should be useful to students and professionals alike as a ready reference to mathematical tools and results of general applicability in econometrics. The two applications chapters should also prove useful to non-economist professionals who are interested in gaining some understanding of certain topics in econometrics.

New York, New York
May 2000

Phoebus J. Dhrymes

Preface to the Second Edition

The reception of this booklet has encouraged me to prepare a second edition.

The present version is essentially the original, but adds a number of very useful results in terms of inverses and other features of partitioned matrices, a discussion of the singular value decomposition for rectangular matrices, issues of stability for the general linear structural econometric model, and similar topics.

I would like to take this opportunity to express my thanks to many of my students and others for pointing out misprints and incongruities in the first edition.

New York, New York
March 1984

Phoebus J. Dhrymes

Preface to the First Edition

This book began as an Appendix to *Introductory Econometrics*. As it progressed, requirements of consistency and completeness of coverage seemed to make it inordinately long to serve merely as an Appendix, and thus it appears as a work in its own right.

Its purpose is not to give rigorous instruction in mathematics. Rather it aims at filling the gaps in the typical student's or professional's mathematical training, to the extent relevant for the study of econometrics.

Thus, it contains a collection of mathematical results employed at various stages of *Introductory Econometrics*. More generally, however, it could serve as a useful adjunct and reference to students of econometrics, no matter what text is being employed.

In the vast majority of cases, proofs are provided and there is a modicum of verbal discussion of certain mathematical results, the objective being to reinforce the student's understanding of the formalities. In certain instances, however, when proofs are too cumbersome, or complex, or when they are too obvious, they are omitted.

New York, New York
May 1978

Phoebus J. Dhrymes

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Chapter 1

Vectors and Vector Spaces

In nearly all of the discussion in this volume, we deal with the set of **real** numbers. Occasionally, however, we deal with **complex** numbers as well. In order to avoid cumbersome repetition, we shall denote the set we are dealing with by $\mathcal{F}$ and let the context elucidate whether we are speaking of real or complex numbers, or both.

1.1 Complex Numbers

For the sake of completeness, we begin with a brief review of complex numbers, although it is assumed that the reader is at least vaguely familiar with the subject.

A **complex** number, say z, is denoted by

$$z = x + iy,$$

where x and y are **real** numbers and the symbol i is defined by

$$i^2 = -1. \tag{1.1}$$

All other properties of the entity denoted by i are derivable from the basic definition in Eq. (1.1). For example,

$$i^4 = (i^2)(i^2) = (-1)(-1) = 1.$$

Similarly,

$$i^3 = (i^2)(i) = (-1)i = -i,$$

and so on.

It is important for the reader to grasp, and bear in mind, that a complex number is describable in terms of an ordered pair of real numbers.

Let

$$z_j = x_j + iy_j, \qquad j = 1, 2,$$

be two complex numbers. We say

$$z_1 = z_2$$

if and only if

$$x_1 = x_2 \quad \text{and} \quad y_1 = y_2.$$

Operations with complex numbers are as follows.

Addition:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

Multiplication by a real scalar:

$$cz_1 = (cx_1) + i(cy_1).$$

Multiplication of two complex numbers:

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Addition and multiplication are, evidently, associative and commutative; i.e. for complex z_j, j=1,2,3,

$$z_1 + z_2 + z_3 = (z_1 + z_2) + z_3 \quad \text{and} \quad z_1 z_2 z_3 = (z_1 z_2) z_3,$$
$$z_1 + z_2 = z_2 + z_1 \quad \text{and} \quad z_1 z_2 = z_2 z_1.$$

and so on.

The **conjugate** of a complex number z is denoted by $\bar{z}$ and is defined by

$$\bar{z} = x - iy.$$

Associated with each complex number is its **modulus** or **length** or **absolute value**, which is a **real** number often denoted by $|z|$ and defined by

$$|z| = (z\bar{z})^{1/2} = (x^2 + y^2)^{1/2}.$$

For the purpose of carrying out multiplication and division (an operation which we have not, as yet, defined) of complex numbers, it is convenient to express them in polar form.

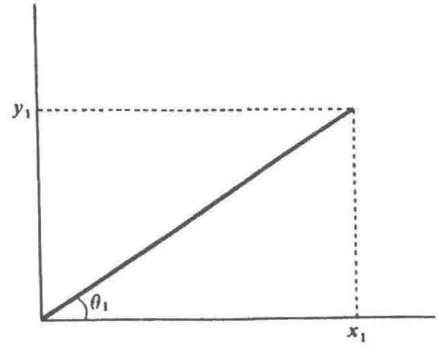

Figure 1.1.

1.1.1 Polar Form of Complex Numbers

Let z, a complex number, be represented in Figure 1 by the point (x_1, y_1), its coordinates.

It is easily verified that the length of the line from the origin to the point (x_1, y_1) represents the modulus of z_1, which for convenience we denote by r_1. Let the angle described by this line and the abscissa be denoted by θ_1. As is well known from elementary trigonometry, we have

$$\cos\theta_1 = \frac{x_1}{r_1}, \qquad \sin\theta_1 = \frac{y_1}{r_1}. \tag{1.2}$$

We may write the complex number as

$$z_1 = x_1 + iy_1 = r_1\cos\theta_1 + ir_1\sin\theta_1 = r_1(\cos\theta_1 + i\sin\theta_1).$$

Further, we may define the quantity

$$e^{i\theta_1} = \cos\theta_1 + i\sin\theta_1, \tag{1.3}$$

and thus write the complex number in the standard **polar form**

$$z_1 = r_1 e^{i\theta_1}. \tag{1.4}$$

In the representation above, r_1 is the **modulus** and θ_1 the **argument** of the complex number z_1. It may be shown that the quantity $e^{i\theta_1}$ as defined in Eq. (1.3) has all the properties of real exponentials insofar as the operations of multiplication and division are concerned. If we confine the **argument** of a complex number to the range $[0, 2\pi)$, we have a unique correspondence between the (x, y) coordinates of a complex number and the modulus and argument needed to specify its polar form. Thus, for any complex number z, the representations

$$z = x + iy, \qquad z = re^{i\theta},$$

where

$$r = (x^2 + y^2)^{1/2}, \quad \cos\theta = \frac{x}{r}, \quad \sin\theta = \frac{y}{r},$$

are completely equivalent.

In polar form, multiplication and division of complex numbers are extremely simple operations. Thus,

$$\begin{aligned} z_1 z_2 &= (r_1 r_2) e^{i(\theta_1+\theta_2)} \\ \frac{z_1}{z_2} &= \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}, \end{aligned}$$

provided $z_2 \neq 0$.

We may extend our discussion to **complex vectors**, i.e. ordered n-tuples of complex numbers. Thus

$$z = x + iy$$

is a complex vector, where x and y are n-element (real) vectors (a concept to be defined immediately below). As in the scalar case, two complex vectors z_1, z_2 are equal if and only if

$$x_1 = x_2, \qquad y_1 = y_2,$$

where now x_i, y_i, $i = 1, 2$, are n-element (column) vectors. The complex conjugate of the vector z is given by

$$\bar{z} = x - iy,$$

and the modulus of the complex vector is defined by

$$(z'\bar{z})^{1/2} = [(x + iy)'(x - iy)]^{1/2} = (x'x + y'y)^{1/2},$$

the quantities $x'x$, $y'y$ being ordinary scalar products of two vectors. Addition and multiplication of complex vectors are defined by

$$\begin{aligned} z_1 + z_2 &= (x_1 + x_2) + i(y_1 + y_2), \\ z_1' z_2 &= (x_1' x_2 - y_1' y_2) + i(y_1' x_2 + x_1' y_2), \\ z_1 z_2' &= (x_1 x_2' - y_1 y_2') + i(y_1 x_2' + x_1 y_2'), \end{aligned}$$

where x_i, y_i, $i = 1, 2$, are real n-element column vectors. The notation for example x_1', or y_2' means that the vectors are written in **row form**, rather than the customary column form. Thus, $x_1 x_2'$ is a matrix, while $x_1' x_2$ is a scalar. These concepts (vector, matrix) will be elucidated below. It is somewhat awkward to introduce them now; still, it is best to set forth at the beginning what we need regarding complex numbers.

1.2 Vectors

Definition 1.1. Let $a_i \in \mathcal{F}$, $i = 1, 2, \ldots, n$; then the ordered n-tuple

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is said to be an n**-dimensional vector**.

Remark 1.1. Notice that a scalar is a trivial case of a vector whose dimension is $n = 1$.

Customarily we write vectors as **columns**, so strictly speaking we should use the term **column vectors**. But this is cumbersome and will not be used unless required for clarity.

If the elements of a vector, a_i, $i = 1, 2, \ldots, n$, belong to $\mathcal{F}$, we denote this by writing

$$a \in \mathcal{F}.$$

Definition 1.2. If $a \in \mathcal{F}$ is an n-dimensional column vector, its **transpose** is the n-dimensional **row** vector denoted by

$$a' = (a_1, \quad a_2, \quad a_3, \quad \ldots, \quad a_n).$$

If a, b are two n-dimensional vectors and $a, b \in \mathcal{F}$, we define their **sum** by

$$a + b = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix}.$$

If c is a scalar and $c \in \mathcal{F}$, we define

$$ca = \begin{pmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{pmatrix}.$$

If a, b are two n-dimensional vectors with elements in $\mathcal{F}$, their **inner product** (which is a scalar) is defined by

$$a'b = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$