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（英文版·第2版）

ALGEBRA

Michael Artin

Second Edition

（美）Michael Artin 麻省理工学院 著

机械工业出版社
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Preface

Important though the general concepts and propositions may be with which the modern and industrious passion for axiomatizing and generalizing has presented us, in algebra perhaps more than anywhere else, nevertheless I am convinced that the special problems in all their complexity constitute the stock and core of mathematics, and that to master their difficulties requires on the whole the harder labor.

—Herman Weyl

This book began many years ago in the form of supplementary notes for my algebra classes. I wanted to discuss some concrete topics such as symmetry, linear groups, and quadratic number fields in more detail than the text provided, and to shift the emphasis in group theory from permutation groups to matrix groups. Lattices, another recurring theme, appeared spontaneously.

My hope was that the concrete material would interest the students and that it would make the abstractions more understandable – in short, that they could get farther by learning both at the same time. This worked pretty well. It took me quite a while to decide what to include, but I gradually handed out more notes and eventually began teaching from them without another text. Though this produced a book that is different from most others, the problems I encountered while fitting the parts together caused me many headaches. I can't recommend the method.

There is more emphasis on special topics here than in most algebra books. They tended to expand when the sections were rewritten, because I noticed over the years that, in contrast to abstract concepts, with concrete mathematics students often prefer more to less. As a result, the topics mentioned above have become major parts of the book.

In writing the book, I tried to follow these principles:

1. The basic examples should precede the abstract definitions.

2. Technical points should be presented only if they are used elsewhere in the book.

3. All topics should be important for the average mathematician.

Although these principles may sound like motherhood and the flag, I found it useful to have them stated explicitly. They are, of course, violated here and there.

The chapters are organized in the order in which I usually teach a course, with linear algebra, group theory, and geometry making up the first semester. Rings are first introduced in Chapter 11, though that chapter is logically independent of many earlier ones. I chose this arrangement to emphasize the connections of algebra with geometry at the start, and because, overall, the material in the first chapters is the most important for people in other fields. The first half of the book doesn't emphasize arithmetic, but this is made up for in the later chapters.

About This Second Edition

The text has been rewritten extensively, incorporating suggestions by many people as well as the experience of teaching from it for 20 years. I have distributed revised sections to my class all along, and for the past two years the preliminary versions have been used as texts. As a result, I've received many valuable suggestions from the students. The overall organization of the book remains unchanged, though I did split two chapters that seemed long.

There are a few new items. None are lengthy, and they are balanced by cuts made elsewhere. Some of the new items are an early presentation of Jordan form (Chapter 4), a short section on continuity arguments (Chapter 5), a proof that the alternating groups are simple (Chapter 7), short discussions of spheres (Chapter 9), product rings (Chapter 11), computer methods for factoring polynomials and Cauchy's Theorem bounding the roots of a polynomial (Chapter 12), and a proof of the Splitting Theorem based on symmetric functions (Chapter 16). I've also added a number of nice exercises. But the book is long enough, so I've tried to resist the temptation to add material.

NOTES FOR THE TEACHER

This book is designed to allow you to choose among the topics. Don't try to cover the book, but do include some of the interesting special topics such as symmetry of plane figures, the geometry of SU_2, or the arithmetic of imaginary quadratic number fields. If you don't want to discuss such things in your course, then this is not the book for you.

There are relatively few prerequisites. Students should be familiar with calculus, the basic properties of the complex numbers, and mathematical induction. An acquaintance with proofs is obviously useful. The concepts from topology that are used in Chapter 9, Linear Groups, should not be regarded as prerequisites.

I recommend that you pay attention to concrete examples, especially throughout the early chapters. This is very important for the students who come to the course without a clear idea of what constitutes a proof.

One could spend an entire semester on the first five chapters, but since the real fun starts with symmetry in Chapter 6, that would defeat the purpose of the book. Try to get to Chapter 6 as soon as possible, so that it can be done at a leisurely pace. In spite of its immediate appeal, symmetry isn't an easy topic. It is easy to be carried away and leave the students behind.

These days most of the students in my classes are familiar with matrix operations and modular arithmetic when they arrive. I've not been discussing the first chapter on matrices in class, though I do assign problems from that chapter. Here are some suggestions for Chapter 2, Groups.

1. Treat the abstract material with a light touch. You can have another go at it in Chapters 6 and 7.

2. For examples, concentrate on matrix groups. Examples from symmetry are best deferred to Chapter 6.
3. Don't spend much time on arithmetic; its natural place in this book is in Chapters 12 and 13.
4. De-emphasize the quotient group construction.

Quotient groups present a pedagogical problem. While their construction is conceptually difficult, the quotient is readily presented as the image of a homomorphism in most elementary examples, and then it does not require an abstract definition. Modular arithmetic is about the only convincing example for which this is not the case. And since the integers modulo n form a ring, modular arithmetic isn't the ideal motivating example for quotients of groups. The first serious use of quotient groups comes when generators and relations are discussed in Chapter 7. I deferred the treatment of quotients to that point in early drafts of the book, but, fearing the outrage of the algebra community, I eventually moved it to Chapter 2. If you don't plan to discuss generators and relations for groups in your course, then you can defer an in-depth treatment of quotients to Chapter 11, Rings, where they play a central role, and where modular arithmetic becomes a prime motivating example.

In Chapter 3, Vector Spaces, I've tried to set up the computations with bases in such a way that the students won't have trouble keeping the indices straight. Since the notation is used throughout the book, it may be advisable to adopt it.

The matrix exponential that is defined in Chapter 5 is used in the description of one-parameter groups in Chapter 10, so if you plan to include one-parameter groups, you will need to discuss the matrix exponential at some point. But you must resist the temptation to give differential equations their due. You will be forgiven because you are teaching algebra.

Except for its first two sections, Chapter 7, again on groups, contains optional material. A section on the Todd-Coxeter algorithm is included to justify the discussion of generators and relations, which is pretty useless without it. It is fun, too.

There is nothing unusual in Chapter 8, on bilinear forms. I haven't overcome the main pedagogical problem with this topic – that there are too many variations on the same theme, but have tried to keep the discussion short by concentrating on the real and complex cases.

In the chapter on linear groups, Chapter 9, plan to spend time on the geometry of SU_2. My students complained about that chapter every year until I expanded the section on SU_2, after which they began asking for supplementary reading, wanting to learn more. Many of our students aren't familiar with the concepts from topology when they take the course, but I've found that the problems caused by the students' lack of familiarity can be managed. Indeed, this is a good place for them to get an idea of a manifold.

I resisted including group representations, Chapter 10, for a number of years, on the grounds that it is too hard. But students often requested it, and I kept asking myself: If the chemists can teach it, why can't we? Eventually the internal logic of the book won out and group representations went in. As a dividend, hermitian forms got an application.

You may find the discussion of quadratic number fields in Chapter 13 too long for a general algebra course. With this possibility in mind, I've arranged the material so that the end of Section 13.4, on ideal factorization, is a natural stopping point.

It seemed to me that one should mention the most important examples of fields in a beginning algebra course, so I put a discussion of function fields into Chapter 15. There is always the question of whether or not Galois theory should be presented in an undergraduate course, but as a culmination of the discussion of symmetry, it belongs here.

Some of the harder exercises are marked with an asterisk.

Though I've taught algebra for years, various aspects of this book remain experimental, and I would be very grateful for critical comments and suggestions from the people who use it.

ACKNOWLEDGMENTS

Mainly, I want to thank the students who have been in my classes over the years for making them so exciting. Many of you will recognize your own contributions, and I hope that you will forgive me for not naming you individually.

Acknowledgments for the First Edition

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The figures which needed mathematical precision were made on the computer by George Fann and Bill Schelter. I could not have done them by myself. Many thanks also to Marge Zabierek, who retyped the manuscript annually for about eight years before it was put onto the computer where I could do the revisions myself, and to Mary Roybal for her careful and expert job of editing the manuscript.

I haven't consulted other books very much while writing this one, but the classics by Birkhoff and MacLane and by van der Waerden from which I learned the subject influenced me a great deal, as did Herstein's book, which I used as a text for many years. I also found some good exercises in the books by Noble and by Paley and Weichsel.

Acknowledgments for the Second Edition

Many people have commented on the first edition – a few are mentioned in the text. I'm afraid that I will have forgotten to mention most of you.

I want to thank these people especially: Annette A' Campo and Paolo Maroscia made careful translations of the first edition, and gave me many corrections. Nathaniel Kuhn and James Lepowsky made valuable suggestions. Annette and Nat finally got through my thick skull how one should prove the orthogonality relations.

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And I talk to Gil Strang and Harold Stark often, about everything.

Finally, I want to thank the many MIT undergraduates who read and commented on the revised text and corrected errors. The readers include Nerses Aramyan, Reuben Aronson, Mark Chen, Jeremiah Edwards, Giuliano Giacaglia, Li-Mei Lim, Ana Malagon, Maria Monks, and Charmaine Sia. I came to rely heavily on them, especially on Nerses, Li-Mei, and Charmaine.

"One, two, three, five, four. . ."
"No Daddy, it's one, two, three, four, five."
"Well if I want to say one, two, three, five, four, why can't I?"
"That's not how it goes."

—Carolyn Artin

Contents

CHAPTER 1

Matrices

Erstlich wird alles dasjenige eine Größe genennt,
welches einer Vermehrung oder einer Verminderung fähig ist,
oder wozu sich noch etwas hinzusetzen oder davon wegnehmen läßt.

—Leonhard Euler[1]

Matrices play a central role in this book. They form an important part of the theory, and many concrete examples are based on them. Therefore it is essential to develop facility in matrix manipulation. Since matrices pervade mathematics, the techniques you will need are sure to be useful elsewhere.

1.1 THE BASIC OPERATIONS

Let m and n be positive integers. An $m \times n$ *matrix* is a collection of mn numbers arranged in a rectangular array

(1.1.1)
$$m \text{ rows} \overset{n \text{ columns}}{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}$$

For example, $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix}$ is a 2×3 matrix (two rows and three columns). We usually introduce a symbol such as A to denote a matrix.

The numbers in a matrix are the *matrix entries*. They may be denoted by a_{ij}, where i and j are indices (integers) with $1 \le i \le m$ and $1 \le j \le n$, the index i is the *row index*, and j is the *column index*. So a_{ij} is the entry that appears in the ith row and jth column of the matrix:

$$\begin{array}{c} \\ i \end{array} \overset{\displaystyle j}{\begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{bmatrix}}$$

[1] This is the opening sentence of Euler's book *Algebra*, which was published in St. Petersburg in 1770.

In the above example, $a_{11} = 2, a_{13} = 0$, and $a_{23} = 5$. We sometimes denote the matrix whose entries are a_{ij} by (a_{ij}).

An $n \times n$ matrix is called a *square* matrix. A 1×1 matrix $[a]$ contains a single number, and we do not distinguish such a matrix from its entry.

A $1 \times n$ matrix is an n-dimensional *row vector*. We drop the index i when $m = 1$ and write a row vector as

$$[a_1 \cdots a_n], \quad \text{or as} \quad (a_1, \ldots, a_n).$$

Commas in such a row vector are optional. Similarly, an $m \times 1$ matrix is an

m-dimensional *column vector*:

$$\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}.$$

In most of this book, we won't make a distinction between an n-dimensional column vector and the point of n-dimensional space with the same coordinates. In the few places where the distinction is useful, we will state this clearly.

Addition of matrices is defined in the same way as vector addition. Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. Their sum $A + B$ is the $m \times n$ matrix $S = (s_{ij})$ defined by

$$s_{ij} = a_{ij} + b_{ij}.$$

Thus

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 3 \\ 4 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 3 \\ 5 & 0 & 6 \end{bmatrix}.$$

Addition is defined only when the matrices to be added have the same shape – when they are $m \times n$ matrices with the same m and n.

Scalar multiplication of a matrix by a number is also defined as with vectors. The result of multiplying an $m \times n$ matrix A by a number c is another $m \times n$ matrix $B = (b_{ij})$, where $b_{ij} = ca_{ij}$ for all i, j. Thus

$$2\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 0 \\ 2 & 6 & 10 \end{bmatrix}.$$

Numbers will also be referred to as *scalars*. Let's assume for now that the scalars are real numbers. In later chapters other scalars will appear. Just keep in mind that, except for occasional reference to the geometry of real two- or three-dimensional space, everything in this chapter continues to hold when the scalars are complex numbers.

The complicated operation is *matrix multiplication*. The first case to learn is the product AB of a row vector A and a column vector B, which is defined when both are the same size, say m. If the entries of A and B are denoted by a_i and b_i, respectively, the product AB is the 1×1 matrix, or scalar,

$$a_1 b_1 + a_2 b_2 + \cdots + a_m b_m. \tag{1.1.2}$$

Thus

$$\begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = 1 - 3 + 20 = 18.$$

The usefulness of this definition becomes apparent when we regard A and B as vectors that represent indexed quantities. For example, consider a candy bar containing m ingredients. Let a_i denote the number of grams of $(\textit{ingredient})_i$ per bar, and let b_i denote the cost of $(\textit{ingredient})_i$ per gram. The matrix product AB computes the cost per bar:

$$(\text{grams/bar}) \cdot (\text{cost/gram}) = (\text{cost/bar}).$$

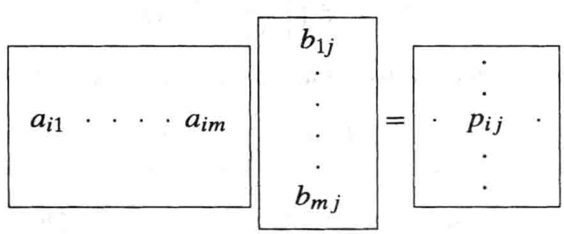
In general, the product of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ is defined when the number of columns of A is equal to the number of rows of B. If A is an $\ell \times m$ matrix and B is an $m \times n$ matrix, then the product will be an $\ell \times n$ matrix. Symbolically,

$$(\ell \times m) \cdot (m \times n) = (\ell \times n).$$

The entries of the product matrix are computed by multiplying all rows of A by all columns of B, using the rule (1.1.2). If we denote the product matrix AB by $P = (p_{ij})$, then

$$p_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{im} b_{mj}. \tag{1.1.3}$$

This is the product of the ith row of A and the jth column of B.



For example,

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}. \tag{1.1.4}$$

This definition of matrix multiplication has turned out to provide a very convenient computational tool. Going back to our candy bar example, suppose that there are ℓ candy bars. We may form the $\ell \times m$ matrix A whose ith row measures the ingredients of $(bar)_i$. If the cost is to be computed each year for n years, we may form the $m \times n$ matrix B whose jth column measures the cost of the ingredients in $(year)_j$. Again, the matrix product $AB = P$ computes cost per bar: p_{ij} = cost of $(bar)_i$ in $(year)_j$.

One reason for matrix notation is to provide a shorthand way of writing linear equations. The system of equations

$$\begin{array}{ccccccc} a_{11}x_1 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

can be written in matrix notation as

(1.1.5) $$AX = B$$

where A denotes the matrix of coefficients, X and B are column vectors, and AX is the matrix product:

$$\boxed{\qquad A \qquad}\begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$

We may refer to an equation of this form simply as an "equation" or as a "system."
The matrix equation

$$\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 18 \end{bmatrix}$$

represents the following system of two equations in three unknowns:

$$\begin{aligned} 2x_1 + x_2 \phantom{{}+5x_3} &= 1 \\ x_1 + 3x_2 + 5x_3 &= 18. \end{aligned}$$

Equation (1.1.4) exhibits one solution, $x_1 = 1$, $x_2 = -1$, $x_3 = 4$. There are others.

The sum (1.1.3) that defines the product matrix can also be written in summation or "sigma" notation as

(1.1.6) $$p_{ij} = \sum_{\nu=1}^{m} a_{i\nu}b_{\nu j} = \sum_{\nu} a_{i\nu}b_{\nu j}.$$