J. S. Lomont

APPLICATIONS OF FINITE GROUPS

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PREFACE

Group theory is primarily a formal mathematical tool for treating symmetry systematically. Consequently group theory is of use to physicists in treating systems of two types: (1) those which possess symmetry but are too complex to be treated in detail by analysis, and (2) those whose properties (e.g. interactions) are not known in detail. It is also useful in formulating theories for which symmetry requirements are specified a priori.

The objectives of this book are: (1) to provide a mathematical background (primarily representation theory) in finite groups which is adequate both for reading this book and for reading the physics literature, and (2) to provide a variety of instructive and interesting examples of applications of finite groups to problems of physics. It is assumed that the reader has been exposed to matrix theory and general group theory. Also a knowledge of quantum mechanics is presupposed. The definitions and theorems (except for the definition of a group) are for finite groups unless otherwise stated.

A chapter on space groups has been included because space groups can also be treated (with a few lapses of rigor) by the methods of finite groups. Also, because finite and continuous groups are rather inextricably mixed in some applications it has been necessary to include brief discussions of some continuous groups. In these discussions, however, the topological aspects of continuous groups were assiduously avoided. The handbook style appendix on Lorentz groups was included as a convenience for research workers.

Chapters 7 and 8 can be read without first reading chapters 5 and 6.

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J. S. LOMONT

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CONTENTS

LIST OF SYMBOLS

*	complex conjugate
t	transpose
†	adjoint or hermitian conjugate or $*t$
$\in$	an element is contained in, e.g., $p \in S$—p is contained in the set S
$\subset$	a subset is properly contained in, e.g., $S_1 \subset S_2$—the set S_1 is properly contained in the set S_2
$\cap$	set-theoretic intersection, e.g., $S_1 \cap S_2$—the set common to the two sets S_1 and S_2
$	summation or integration
$\rightarrow$	is mapped onto
$\leftrightarrow$	corresponds to, e.g., $A \leftrightarrow D$—A corresponds to D in a one-to-one mapping
$\equiv$	is equivalent to
{ }	set of all elements
$\otimes$	Kronecker product
$\oplus$	direct sum

$$[D, D^\dagger] = 0.$$

On the other hand, if $DD^\dagger + D^\dagger D = 0$, then $D = 0$. Some important special types of normal matrices are:

(1) unitary matrices $D^\dagger D = I$
(2) hermitian matrices $D^\dagger = D$
(3) orthogonal matrices $D^t D = I$ $D^* = D$ (We consider only real orthogonal matrices, which are therefore also real unitary matrices.)
(4) rotation matrices (These are orthogonal matrices which have positive determinants.)
(5) symmetric matrices $D^t = D$ $D^* = D$ (Again we consider only real ones.)
(6a) permutation matrices (A permutation matrix is a square matrix whose elements are all either one or zero and which has exactly one nonzero entry in each row and exactly one in each column.)

Example.

$$D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

(6b) pseudopermutation matrices (Let us understand by this term any matrix obtained from any permutation matrix by replacing some of the ones by minus ones.)
(7) diagonal matrices (All off-diagonal elements are zero.)
(8) scalar matrices $D = \chi I$, where χ is any complex number.

Some other special types of matrices which will be of interest are

(1) unimodular (or special) matrices: $\det(D) = 1$
(2) integral matrices (Every element is an integer.)
(3) idempotent matrices $D^2 = D$
(4) nilpotent matrices $D^N = 0$, for some positive integer N
(5) monomial matrices (These are square matrices having only one nonzero entry in each row and only one in each column.)
(6) real matrices $D^* = D$
(7) skew symmetric matrices $D^t = -D$

We can now proceed to the meat of matrix theory. Subsequently, the entry in the ith row and jth column of a matrix D will be denoted by D_{ij}.

Definition. The *trace* (or spur) of a square d-dimensional matrix D (abbreviated $\operatorname{tr}(D)$) is the sum of the diagonal elements of D.

$$\operatorname{tr}(D) \equiv \sum_{l=1}^{d} D_{ll}.$$

Note. If D_1 and D_2 are square matrices of the same dimension, then

$$\operatorname{tr}(D_1 D_2) = \operatorname{tr}(D_2 D_1)$$

Note. If D is a unitary matrix of dimension d, $\operatorname{tr}(D) = \pm d$, then $D = \pm I$ (according to the sign of the trace).

Definition. Let D be a d-dimensional square matrix.

(a) The *characteristic polynomial* of D is $\det(D - \lambda I)$ (which is a dth degree polynomial in λ with leading coefficient $(-1)^d$).

(b) The *characteristic equation* of D is $|D - \lambda I| = 0$.

(c) The *eigenvalues* of D are the d roots $\lambda_1,\ldots,\lambda_d$ of the characteristic equation of D.

It can easily be shown that the eigenvalues of hermitian matrices are real and that the eigenvalues of unitary matrices have modulus one.

Theorem. If D is a d-dimensional square matrix with eigenvalues $\lambda_1,\ldots,\lambda_d$, then

$$\operatorname{tr}(D) = \sum_{i=1}^{d} \lambda_i.$$

Definition. The *degeneracy* of an eigenvalue of a square matrix D is the number of times it occurs as a root of the characteristic equation of D.

Definition. A *positive definite* square matrix is a hermitian matrix whose eigenvalues are all positive.

Theorem.
$$\det(D) = \prod_{i=1}^{d} \lambda_i.$$

Note. (1) A skew symmetric matrix of odd dimension has determinant zero.

(2) The dimension of a skew symmetric unitary matrix cannot be odd.

Corollary. A square matrix is nonsingular (or regular) if and only if it has no eigenvalues equal to zero.

Also, it may be recalled that if all the elements of a matrix are positive, then the matrix has one nondegenerate positive eigenvalue whose magnitude is larger than that of any other eigenvalue.

Definition. Two square matrices D_1 and D_2 such that

$$D_2 = S^{-1} D_1 S,$$

where S is a nonsingular square matrix, are said to be *equivalent* (or similar) (we shall write $D_2 \cdot\!\equiv\!\cdot D_1$).

Note that this implies that D_1 and D_2 must have the same dimension and that $D_1 \cdot\!\equiv\!\cdot D_2$. It may be recalled that an equivalence transformation can be geometrically interpreted as a change of axes (alias transformation) or as a point transformation (alibi transformation). Also, if two equivalent matrices D_1 and D_2 are given, then the problem of finding the general form of the transformation matrix S connecting them is quite laborious. We shall return to this problem later.

Theorem. $D_2 \cdot\equiv\cdot D_1$ implies

(1) $\operatorname{tr}(D_2) = \operatorname{tr}(D_1)$

(2) $\det(D_2) = \det(D_1)$

(3) D_2 and D_1 have the same characteristic polynomial

(4) D_2 and D_1 have the same eigenvalues.

Every square matrix D is equivalent to a simple "almost"-diagonal matrix Λ which depends on D and which is known as the Jordan canonical form of D. However, we shall not discuss this further here.

Definition. A *diagonalizable* matrix is a square matrix which is equivalent to a diagonal matrix.

No simple criterion for diagonalizability seems to be known, but a few useful relevant results are known.

Theorem. The diagonal elements of a diagonal matrix equivalent to D are the eigenvalues of D.

Theorem. A square matrix is diagonalizable if its eigenvalues are non-degenerate.

We shall say that a diagonalizable matrix D is diagonalizable by a unitary matrix if there exists a unitary matrix U such that $U^{-1}DU$ is a diagonal matrix.

Theorem. (1) A square matrix D is diagonalizable by a unitary matrix if and only if D is a normal matrix.

(2) A real square matrix D is diagonalizeable by an orthogonal matrix if and only if D is a symmetric matric.

Hence, the various special types of normal matrices listed earlier are all diagonalizable. In fact, practically all matrices which will be of interest to us will be diagonalizable. We shall now state several well-known equivalence theorems for diagonalizable matrices.

Theorem. Two diagonalizable matrices D_1 and D_2 are equivalent if and only if

$$(1)\quad \dim(D_1) = \dim(D_2) \qquad (= d)$$
$$(2)\quad \operatorname{tr}(D_1^{\,i}) = \operatorname{tr}(D_2^{\,i}) \qquad i = 1,\ldots,d.$$

Theorem. Each of the following is a necessary and sufficient condition for a diagonalizable matrix D to be equivalent to a real matrix.

(1) The eigenvalues of D occur in complex conjugate pairs.

(2) The traces of the first d powers of D are real [where $d = \dim(D)$].

Theorem. A matrix D is equivalent to a real matrix if and only if $D \cdot\equiv\cdot D^*$.

Theorem. A matrix is equivalent only to real matrices if and only if it is a real scalar matrix.

Theorem. A diagonalizable matrix D is equivalent to an integral matrix if and only if (1) the traces of the first d powers of D are integers [$d = \dim(D)$]; (2) the eigenvalues of D are algebraic integers.

By an integer is meant a real whole number, and by an algebraic integer is meant a root of an equation of the form

$$X^N + a_{N-1}X^{N-1} + a_{N-2}X^{N-2} + \dots + a_0 = 0,$$

in which all of the a's are integers. We shall see later that crystallographic groups can be considered to have elements which are integral matrices.

Theorem. If two real matrices are equivalent, then they are equivalent with respect to a real transformation.

That is, if $D_2 = S^{-1}D_1S$, where D_1 and D_2 are real, then there exists a real matrix T such that $D_2 = T^{-1}D_1T$. In fact, if we split S into real and imaginary parts, $S = P + iQ$, and let λ be any real number such that $|P + \lambda Q| \neq 0$, then we can put $T = P + \lambda Q$.

Definition. A *projection* matrix is an idempotent diagonalizable matrix.

We now move on to some more difficult concepts which involve sets of matrices. Although the theorems will be stated for finite sets they hold also for infinite sets.

Definition. Let $D_1, D_2, \dots, D_N$; $D_1', D_2', \dots, D_N'$ be two sets (not necessarily finite) of square matrices. These two *sets* will be said to be *equivalent* if there exists a square matrix S such that

$$D_i' = S^{-1}D_iS \qquad i = 1,\dots,N$$

for some ordering of the second set.

Example.

$$\Gamma_1: \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}-i&0\\0&i\end{pmatrix}\begin{pmatrix}i&0\\0&-i\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0&-i\\-i&0\end{pmatrix}\begin{pmatrix}0&i\\i&0\end{pmatrix}$$

$$\Gamma_2: \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}-1&0\\0&-1\end{pmatrix}\begin{pmatrix}5i&8i\\-3i&-5i\end{pmatrix}\begin{pmatrix}-5i&-8i\\3i&5i\end{pmatrix}\begin{pmatrix}-7&-10\\5&7\end{pmatrix}\begin{pmatrix}7&10\\-5&-7\end{pmatrix}$$
$$\begin{pmatrix}5i&6i\\-4i&-5i\end{pmatrix}\begin{pmatrix}-5i&-6i\\4i&5i\end{pmatrix}$$

where $\Gamma_2 \cdot\!\equiv\!\cdot \Gamma_1$ and $S = \begin{pmatrix}1&2\\3&4\end{pmatrix}$ $\qquad S^{-1} = -\frac{1}{2}\begin{pmatrix}4&-2\\-3&1\end{pmatrix}$.

It follows that all of the matrices D_i, D_i', and S must have the same dimension. Also, it is possible to have

$$D_i' \cdot\!\equiv\!\cdot D_i \qquad i = 1,\dots,N$$

without having the sets equivalent.

Theorem. If two sets of real matrices are equivalent, then they are equivalent with respect to a real transformation.

This result will be essential to our group theoretic derivation of crystallographic point groups.

Theorem. If two sets of unitary (orthogonal) matrices are equivalent, then they are equivalent with respect to a unitary (orthogonal) transformation.

Theorem. If a set of unitary matrices is equivalent to a set of real matrices, then it is equivalent to a set of orthogonal matrices.

Definition. A set of matrices will be said to be diagonalizable if the set is equivalent to a set of diagonalizable matrices.

Note that a set of diagonalizable matrices need not be a diagonalizable set of matrices. In fact, there is a very simple result for normal matrices.

Theorem. A set of normal matrices of the same dimension is diagonalizable if, and only if, the commutator of every pair of matrices of the set is zero.

Such a set of matrices is called a commuting set, and one sees immediately that every set of diagonal matrices is a commuting set.

Definition. (a) A set of d-dimensional diagonal matrices $\Lambda_1, \ldots, \Lambda_N$ is *complete* if no two sets S_j and S_k of the d sets of eigenvalues $S_i = (\Lambda_{1,ii}; \Lambda_{2,ii}; \ldots; \Lambda_{N,ii})$ are equal term by term. (A single matrix with no degenerate eigenvalues is thus a complete set.)

Example.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

is a complete set because

$$S_1 = (1,1,2), \quad S_2 = (1,2,1), \quad S_3 = (2,2,1), \quad S_4 = (2,1,2),$$

and no two of these sets are equal term by term.

(b) A *complete commuting set* of normal matrices is a set of normal matrices which is equivalent to a complete set of diagonal matrices.

This concept is important in labelling states in the Dirac formulation of quantum mechanics.

We proceed now to the concept of reducibility and represent subarrays of a matrix by matrices. Thus we shall consider matrices whose elements are also matrices.

Definition. A *reduced* matrix is a matrix of the form

$$\begin{pmatrix} D(11) & 0 \\ D(21) & D(22) \end{pmatrix} \text{ or } \begin{pmatrix} D(11) & D(12) \\ 0 & D(22) \end{pmatrix},$$

where $D(ij)$ is a matrix, $D(ii)$ is square, and the off-diagonal zero matrix has at least one row.

Example.

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 6 & 7 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 7 \\ 0 & 0 & 5 \end{pmatrix}$$

are reduced matrices.

In fact, since the Jordan canonical form of a matrix is reduced, it follows that every matrix is equivalent to a reduced matrix, or is reducible (if $d \neq 1$).

Definition. A *set* of matrices $D_1, \ldots, D_N$ (of the same dimension) is ***reduced*** if every matrix of the set has the reduced form

$$D_i = \begin{pmatrix} D_i(11) & 0 \\ D_i(21) & D_i(22) \end{pmatrix},$$

where $\dim D_i(11) = \dim D_j(11)$, $i, j = 1, \ldots, N$, and the zero matrices have at least one row (or, the zero matrix may be in the lower left corner of all D's instead of the upper right).

Reducibility is thus a weak substitute for diagonalizability.

Example.

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 6 & 7 & 5 \end{pmatrix} \quad \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 6 & 5 & 7 \end{pmatrix}$$

is a reduced set.

We finally arrive at the concept of reducibility of a set.

Definition. A *set* of matrices is ***reducible*** if it is equivalent to a reduced set of matrices.

Reducibility is thus a sort of weakened diagonalizability, because every diagonalizable matrix is reducible although the converse is not true. Also, we have seen that every matrix is reducible. However, not every set of matrices is reducible. For example, the set

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

is not reducible. Also, any set of 1-dimensional matrices is not reducible.

Definition. A set of square matrices of the same dimension is ***irreducible*** if it is not reducible.

We come now to a result which is the key to the proofs of many results of representation theory, and which is known as Schur's lemma.

Lemma (Schur). Let $D_1', \ldots, D_N'$ be an irreducible set of d'-dimensional matrices; and $D_1'', \ldots, D_N''$ an irreducible set of d''-dimensional matrices. Then, if there exists a matrix S such that $D_i'S = SD_i''$ for some ordering of the second set, then it follows that either

(1) S is a zero matrix, or (2) S is a square nonsingular matrix (so $d' = d''$).

Outline of Proof. Let the d'' columns of S be denoted by $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \ldots, \boldsymbol{\sigma}_{d''}$. Then by the rules of matrix multiplication we find that for a typical matrix D' and a typical matrix D''

$$D'S = (D\boldsymbol{\sigma}'_1, D\boldsymbol{\sigma}'_2, \ldots, D\boldsymbol{\sigma}'_{d'})$$

$$SD'' = \left(\sum_{k=1}^{d''} D''_{k1}\boldsymbol{\sigma}_k, \sum_{k=1}^{d''} D''_{k2}\boldsymbol{\sigma}_k, \ldots, \sum_{k=1}^{d''} D''_{kd''}\boldsymbol{\sigma}_k\right).$$

Hence

$$D'\boldsymbol{\sigma}_j = \sum_{k=1}^{d''} D''_{kj}\boldsymbol{\sigma}_k$$

and we see that the d'' $\boldsymbol{\sigma}$-vectors span a space which is invariant under the irreducible set of d'-dimensional matrices $\{D'\}$. Consequently, the $\boldsymbol{\sigma}$-vectors are the null vector or they span a d'-dimensional vector space. In the first case $S = 0$, and in the second case $d'' \geqslant d'$ and $S \neq 0$.

Let us consider the second case further. From the fact that the sets $D_1', \ldots, D_N'$ and $D_1'', \ldots, D_N''$ are irreducible it easily follows that the sets $D_1^\dagger, \ldots, D_N^\dagger$ and $D_1''^\dagger, \ldots, D_N''^\dagger$ are irreducible. Also, from the equations $D_i'S = SD_i''$ it follows directly that $D_i''^\dagger S^\dagger = S^\dagger D_i'^\dagger$. Applying the procedure of the preceding paragraph to these equations we find that $d' \geqslant d''$. Hence, $d' = d''$, and S is square. Also, since the d' columns of S span a d'-dimensional vector space, it follows that S is nonsingular.

Corollary. A matrix D which commutes with an irreducible set of matrices $D_1, \ldots, D_N$ (i.e., $[D, D_i] = 0$, $i = 1, \ldots, N$) must be a scalar matrix.

Corollary. If S_1 and S_2 are two matrices such that $D_i'S_j = S_jD_i''$ and the two sets $\{D_i'\}$ and $\{D_i''\}$ are irreducible, then $S_2 = \lambda S_1$ where λ is a number.

Theorem. Let $D_1, \ldots, D_N$ be an irreducible set of d-dimensional matrices. Then every d-dimensional matrix can be expressed as a polynomial in the matrices $D_1, \ldots, D_N$.

We shall say that a set of square matrices is equivalent to its complex conjugate if the two sets are not only equivalent in the usual sense but are also equivalent in the same order; i.e., if there exists a matrix S such that $D_i^* = S^{-1}D_iS$.

Theorem. Let $\{D_1, \ldots, D_N\}$ be an irreducible set of square unitary matrices of the same dimension which is equivalent to its complex conjugate set (with the same order). Let U be any unitary matrix such that $D_i^* = U^{-1}D_iU$ for all D_i. Then, (1) if $\{D_i\}$ is equivalent to a real set of matrices, $U^t = U$, (2) otherwise, $U^t = -U$.

We come now to the concepts of direct sum and complete reducibility.

Definition. (a) Let D, D', D'' be three square matrices. Then D is the *direct sum* of D' and D'' if

$$D = \begin{pmatrix} D' & 0 \\ 0 & D'' \end{pmatrix}$$

or

$$D = \begin{pmatrix} D'' & 0 \\ 0 & D' \end{pmatrix}$$

This is written $D = D' \oplus D''$.

(b) Let $\Gamma = \{D_1,\dots,D_N\}$, $\Gamma' = \{D_1',\dots,D'_N\}$, $\Gamma'' = \{D_1'',\dots,D''_N\}$ be three sets of matrices. Then Γ is the direct sum of Γ' and Γ'' (written $\Gamma = \Gamma' \oplus \Gamma''$) if

$$D_i = \begin{pmatrix} D_i' & 0 \\ 0 & D_i'' \end{pmatrix}$$

or if

$$D_i = \begin{pmatrix} D_i'' & 0 \\ 0 & D_i' \end{pmatrix}.$$

Example.

$$\Gamma = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right\}, \Gamma' = \{1, -1\}, \Gamma'' = \{1,1\}.$$

Definition. A set of matrices is *completely reducible* if it is equivalent to the direct sum of two other sets of matrices (written $\Gamma \cdot\!\equiv\!\cdot \Gamma' \oplus \Gamma''$).

If a set is itself the direct sum of two sets, then it is completely reducible (as is Γ of the preceding example).

Note. If Γ is completely reducible it is reducible.

This statement is true because complete reducibility simply means that there is a zero matrix in both the upper right and lower left corners of the completely reduced matrix; whereas reducibility requires only one such zero matrix. The converse of this theorem is not true.

Theorem. $\Gamma \cdot\!\equiv\!\cdot \Gamma' \oplus \Gamma''$ implies $\dim(D_i) = \dim(D_i') + \dim(D_i'')$

$$\operatorname{tr}(D_i) = \operatorname{tr}(D_i') + \operatorname{tr}(D_i'')$$

Note. If the only matrices which commute with a set of square matrices of the same dimension are scalar matrices, then the set of matrices is not completely reducible.

One of our preceding theorems can now be generalized slightly.

Theorem. Let $\{D_1,\dots, D_N\}$ be a set of square unitary matrices of the same dimension which is equivalent to its complex conjugate set (in the same order), but not to a set of real matrices. Also let the set be completely reduced into irreducible unitary components, and let U be any unitary matrix such that $D_i^* = U^{-1}D_iU$. Then $U^t = -U$.

One can form not only direct sums of matrices but also direct products.

Definition. The Kronecker product (or direct product) of two square matrices D' and D'' is the matrix D whose element in the ijth row and klth column is given by

$$D_{ij,kl} = D'_{ik} D''_{jl}$$

(written $D = D' \otimes D''$). (The actual ordering of rows in D is not important for our work and consequently will not be specified.)

Theorem. (1) $\dim (D' \otimes D'') = \dim (D') \times \dim (D'')$

(2) $\mathrm{tr}\,(D' \otimes D'') = \mathrm{tr}\,(D') \times \mathrm{tr}\,(D'')$

(3) the eigenvalues of $D' \otimes D''$ are the products $\lambda_i' \lambda_j''$ where λ_i' and λ_j'' are eigenvalues of D' and D''.

Theorem. If $\dim D_1' = \dim D_2'$
$\dim D_1'' = \dim D_2''$,

then

$$(D_1' \otimes D_1'')(D_2' \otimes D_2'') = (D_1'D_2') \otimes (D_1''D_2'').$$

Theorem. For every D', D'' there exists a $d'd''$-dimensional permutation matrix P such that $D' \otimes D'' = P^{-1}(D'' \otimes D')P$.

Definition. If $\Gamma' = \{D_1', \dots, D'_{N'}\}$ is a set of d'-dimensional matrices and $\Gamma'' = \{D_1'', \dots, D''_{N''}\}$ is a set of d''-dimensional matrices, then

(a) $\Gamma' \otimes \Gamma'' = \{D_1' \otimes D_1'', \dots, D'_{N'} \otimes D''_{N''}\}$ if $N' = N''$

(b) $\Gamma' \times \Gamma''$ = the set of all $N'N''$ matrices $D_i' \otimes D_j''$

$$i = 1, \dots, N' \qquad j = 1, \dots, N''$$

are called the *inner* and *outer* Kronecker products of the two sets.

If D is a square, d-dimensional matrix, then let us call the set of all d-dimensional column matrices the *carrier space* of D. This is obviously a d-dimensional vector space. Furthermore, if V is a vector of the carrier space of D, one can define its transformed vector V' by

$$v' = Dv$$

or

$$v_i' = D_{ij} v_j$$

(using the summation convention for repeated indices). Then, a kth rank tensor $F_{i_1, \dots, i_k}$ which transforms under D will be transformed as a product of vectors,

$$F'_{i_1 \dots i_k} = D_{i_1 j_1} D_{i_2 j_2} \dots D_{i_k j_k} F_{j_1 \dots j_k}.$$

That is, a kth rank tensor is transformed by the kth Kronecker power of D (call it D_k), or, then tensor F is a vector in the carrier space of D_k.

If one considers the transformation properties of the independent