STUDIES IN APPLIED MATHEMATICS 5

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ADVANCES IN DIFFERENTIAL AND INTEGRAL EQUATIONS

A collection of papers presented at the Conference on Qualitative Theory of Nonlinear Differential and Integral Equations, sponsored by the University of Wisconsin and Society for Industrial and Applied Mathematics, supported by the Office of Naval Research and held at the University of Wisconsin.

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Preface

This volume consists of invited lectures, together with abstracts of contributed papers, delivered at a conference on Qualitative Theory of Nonlinear Differential and Integral Equations, held at the University of Wisconsin in Madison, August 19–23, 1968, under the sponsorship of the University and the Society for Industrial and Applied Mathematics, and supported by a grant from the Office of Naval Research.

As indicated by the table of contents, the lectures divide naturally into five areas; some of the papers overlap several fields. The abstracts of contributed papers have been appended to the chapters in accordance with the topic. Several of the contributed papers are very closely related to particular longer papers discussed in the main part of the chapter. Two lectures were presented at the Conference in addition to the papers appearing here. Professor Jürgen Moser of the Courant Institute of Mathematical Sciences presented a new formulation of Sacker's result on continuation under perturbations of smooth invariant manifolds. Professor Stephen Smale, University of California at Berkeley, spoke on "Global stability questions," concerning diffeomorphisms of a smooth, compact manifold; he discussed necessary and sufficient conditions for structural as well as Ω-stability. His lecture will appear in the Proceedings of the 1968 AMS Summer Institute in Global Analysis.

It would, of course, be an impossible task to provide in a single volume an account of current research in nonlinear differential and integral equations. Nevertheless, acknowledging some regrettable omissions, these lectures represent a considerable amount of the research activity in this general area. We can best indicate the scope of the Conference by giving a brief description of the content of each lecture; we do this in the order in which these papers appear in the text.

Partial differential equations of elliptic and parabolic type of second order with a divergence structure play an important role in both theory and applications. D. G. ARONSON discusses a rather complete theory concerning the behavior of weak solutions of nondegenerate quasilinear parabolic problems (maximum principle, local boundedness results, Harnack-type inequality, etc.). Such techniques generally do not apply in the degenerate case. This is explained for one specific equation and several open questions are described.

Eigenvalue problems for operators of the form

$$\lambda^q I - A - \sum_{k=0}^{n} \lambda^{p+q} B_k,$$

where λ is a complex number, $p \geqq 0$, $q > 0$ are integers, and where $A, B_0, \cdots, B_n$ are compact operators on a Hilbert space H, arise for certain elliptic boundary value problems. A. FRIEDMAN discusses the problem of completeness of "generalized eigenvectors" of such operators under broad hypotheses involving A and the B_k.

As was established recently by P. D. Lax, for a single scalar equation, the existence of waves obeying a nonlinear wave equation for all past and future times cannot be expected. The plane elastic waves of small finite amplitude, studied by F. JOHN, lead to an investigation of a quasilinear hyperbolic system for which he discusses traveling waves, longitudinal and transverse waves, and questions of global existence and nonexistence of waves requiring considerable extensions of Lax's result.

Using new a priori estimates for linear equations (similar to known estimates for the L^2-theory of elliptic boundary value problems) and various iteration schemes (including a rapid convergence method due to J. Moser), P. RABINOWITZ discusses the existence, uniqueness and stability of periodic solutions of nonlinear hyperbolic problems of the form $Lu + \varepsilon F = 0$. Here L is a second order linear hyperbolic operator, ε is a small parameter, and the nonlinear function F, which is periodic in t, also depends on the space variables x and on u and its derivatives up to order two.

Higher order Dirichlet elliptic problems and initial Dirichlet parabolic problems have been the subject of recent studies whose aim has been to weaken, as much as possible, the hypotheses concerning both the coefficients and the data. E. FABES and N. RIVIÈRE obtain L^p-estimates ($1 < p \leqq \infty$) which, in the parabolic case with Hölder coefficients, lead to new existence and uniqueness theorems that only require the boundary data to be of class L^p. In the elliptic case these results generalize estimates obtained by Agmon.

An important question in boundary layer theory is the existence of classes of similarity solutions of the underlying systems of quasilinear partial differential equations. J. B. SERRIN discusses two such problems, the flow of a compressible fluid past a fixed wall and free thermal convection of a viscous fluid adjacent to a heated vertical plate. Both problems lead to new questions concerning the existence of a solution of a boundary value problem for systems of nonlinear ordinary differential equations on the infinite interval. The method of proof is essentially constructive in nature and can serve as a theoretical basis for making useful numerical calculations. These techniques may be employed to analyze other problems, e.g., swirling flow generated by a revolving plane disk.

Massera and Shäffer have investigated admissibility with respect to integral operators generated by solutions of linear systems of ordinary differential equations. CORDUNEANU surveys recent extensions of these ideas to the study of admissibility of a pair of function spaces (E, F) with respect to an integral operator

K defined by $(Kx)(t) = \int_{-\infty}^{\infty} k(t, s)x(s)\, ds$, where $x \in E$, and $k(t, s)$ (usually a matrix)

is at least measurable. He also discusses the applications to the study of various qualitative properties of solutions of nonlinear functional-integral equations whose kernel is $k(t, s)$.

Processes of learning based on psychological postulates can be described by systems of nonlinear difference-differential equations. Their solutions describe cross-correlated flows over probabilistic networks. S. GROSSBERG establishes global limit and oscillation theorems for these systems and uses them to obtain a theory of learning, remembering and recall by the network of spatial patterns presented to it by an experimentalist as an input.

There exists a rather complete qualitative theory for functional differential equations (equations with retarded argument). For example, in the autonomous case, and in the case of an elementary critical point, it is known that the stable and unstable manifold near such points are essentially the same as those of the linear approximation. J. K. HALE discusses these and analogous questions for the behavior near a periodic orbit; sufficient conditions are given for the existence of a periodic orbit of a perturbed equation when the unperturbed equation possesses a nondegenerate periodic orbit. As might be expected from the ordinary differential equation analogue, this problem presents serious difficulties.

Under hypotheses motivated by various applications, J. J. LEVIN obtains boundedness and oscillation theorems for solutions of the nonlinear Volterra equation

$$x'(t) = -\int_0^t g(x(t-\tau))\,dB(\tau) + f(t),$$

where B is a nonnegative normalized function of bounded variation, g is a given nonlinear function, and f is a given perturbation. Extensions to other types of delay equations are given. Boundedness theorems previously obtained (e.g., by energy methods) are much more restrictive than the present ones and yield as a special case results for linear equations. One important point is that the present main boundedness result depends very much on the type of nonlinearities considered and specifically excludes the linear case.

Topological dynamics has had important applications to the theory of autonomous and period systems of ordinary differential equations. R. K. MILLER shows that solutions of Volterra integral equations of the form

$$(*) \qquad x(t) = f(t) + \int_0^t g(t, s, x(s))\,ds, \qquad t \geqq 0,$$

exhibit the structure of a dynamical system. Since solutions need not be defined for $t < 0$, one now speaks of semiflows rather than flows. Roughly speaking, with $f \in C[0, \infty)$ and g restricted to a suitable function space G, the map $\pi(t; f, g)$ with values in $C[0, \infty) \times G$ exhibits the usual properties of a dynamical system: $\pi(0; f, g) = f(g)$, $\pi(t; \pi(s; f, g)) = \pi(t + s; f, g)$ and π is continuous in the triple (t, f, g). These ideas are made precise and established in a general setting. Numerous known results for behavior of solutions of (*) are, in special cases, reformulated and deduced from the general theory.

The study of closed invariant sets of a smooth flow on a compact manifold M has many applications. CONLEY and EASTON consider the collection of such sets as a subspace of all closed subsets of M and single out the so-called isolated and quasi-isolated invariant sets. They introduce the notion of an isolating block in order to describe homologically an individual isolated invariant set. Numerous examples and open questions are discussed.

An essay by S. DILIBERTO considers, from the Poincaré point of view, the central problem for classical dynamical systems: the three-body problem. He discusses and criticizes the (mathematical) successes and limitations of approaches that have resulted from efforts to tackle this unsolved problem of long standing, as well as the simpler (and yet not completely solved) restricted three-body and Hill problems. In the final section he outlines a new formal approach to the problem of stability of the orbit of the moon. A rigorous convergence proof of this technique which would give an affirmative solution to the main problem awaits further research.

The qualitative behavior of solutions of the autonomous system $\dot{x} = f(x)$, where $f \in C^r(R^n)$, in some tubular neighborhood of a periodic solution $(S) x = \varphi(t)$ of least period $\tau > 0$ is conveniently studied by means of the "transversal germ" of (S) (and in some cases by means of its differential at the origin). L. MARKUS uses this and the concept of C^r-equivalence of periodic orbits to classify the periodic orbits as to topological type and to discuss what happens under C^r-small perturbations of f. Special attention is devoted to analytic systems; for these the results show that even if (S) is not elementary (some of the multipliers are 1) there exists a tubular neighborhood N around (S) such that every other periodic solution encircling N just once, lies on some analytic 2-manifold filled by such periodic orbits. In R^3 the results obtainable are stronger.

Consider a vector field on a compact two-dimensional manifold. Assume these are a finite number of nondegenerate singularities, no attracting singularities and no closed curves joining saddles. R. SACKER and G. SELL give sufficient conditions for the existence of a nontrivial recurrent motion (one which is not a singularity). The manner in which detractors (attractors with all vectors reversed) are joined to the saddles by asymptotic orbits plays a decisive role. Several examples are given.

Consider the system $x' = f(t, x)$, where $f \in C(R \times Z^n; Z^n)$, the set of all continuous functions from $R \times Z^n$ into Z^n, where R is the set of reals and Z^n is complex Euclidean n-space. Consider the following subsets of $C(R \times Z^n; Z^n)$: AP (the set of almost-periodic functions), CAA (the set of compact almost-automorphic functions), and RE (the set recurrent functions). G. SEIFERT discusses these sets of functions and the following question: Let $f \in AP$ or CAA or RE. Let φ be a solution which lies in a compact set $K \subset Z^n$ for all $t \in R$. Under what conditions is it true that, respectively, $\varphi \in AP$, $\varphi \in CAA$, or $\varphi \in RE$? A number of open questions are presented.

Several mathematical questions arise in the problem of retrieval of a derelict spaceship (with no evasive capability) drifting through the gravitational field of the solar system. If it is assumed that a pursuit ship has a controlling thrust, limited in magnitude but unlimited in time duration, L. MARKUS and G. SELL prove that capture is always possible. If the derelict spaceship has some evasive capability, the problem is open. Another open question arises if certain periodicity or almost-periodicity requirements are dropped.

General stability results for autonomous systems of differential equations are discussed by G. O. SZEGO. He first discusses flows without uniqueness and then obtains extension theorems for such flows.

The problem of determining criteria for the existence of periodic solutions for nonlinear ordinary differential equations, delay-differential equations, and hyperbolic partial differential equations has been studied extensively. L. CESARI presents a unifying abstract approach to these questions by examining operator equations of the form $Kz = 0$ with solutions to be found in a given part G of a Banach space B. Extensive applications of the theory are given.

Suppose that a nonlinear system (**) $x' = f(t, x)$ is invariant under a transformation of variables T given by the equations $s = P(t, x)$, $y = h(t, x)$, where P is a scalar and h is a vector. Under suitable regularity conditions on f, p, h, if $x(t)$ is a solution of (**), there is a companion solution $y(t)$ such that $y(P(t, x(t))) = h(t, x(t))$. D. C. LEWIS applies such transformations T which are "self-companiate" ($y(t) = x(t)$) to prove existence of periodic solutions of systems of the form (**) which are invariant under suitable transformations T.

The study of the behavior of periodic solutions of the scalar equation $x'' + g(x) = Ef(t)$, where E is a constant, $g(x)$ is odd, $f(t)$ is 2π-periodic, even- and odd-harmonic ($f(t + A) - f(t)$), is important in several applications. W. LOUD gives a development of the theory for the existence of a 2π-periodic solution which is odd-harmonic. If the range of $g'(x)$ includes an even square integer, 2π-periodic solutions which are not odd-harmonic may exist. Branching and a jump phenomenon which can occur in such cases are discussed.

The existence of periodic and almost-periodic solutions of nonlinear systems is most often established by using the existence of a bounded solution having certain stability properties. All recent studies employed the theory of dynamical systems and hence uniqueness is required in the technique. T. YOSHIZAWA introduces the notion of the Fréchet asymptotic almost-periodic function and he shows that for *A.-P.* systems all results obtained via dynamical systems can be obtained directly from this function.

Set-valued mappings play an important role whenever solutions of nonlinear differential equations are required to lie in a preassigned convex set of continuous functions. H. ANTOCIEWICZ surveys recent results in this field by several authors and gives numerous applications. Further results on such problems, including comparison theorems, are given by H. HERMES.

Consider the nonlinear system $x^\sigma(dy/dx) = f(x, y)$, where x is a complex variable y, f are n-vectors, with f holomorphic at the origin, and the matrix $x^\sigma = \operatorname{diag}(x^{\sigma_1}, x^{\sigma_2}, \cdots, x^{\sigma_n})$, where σ_i are nonnegative integers. W. A. HARRIS, JR. examines the problem of existence of a solution holomorphic at 0 by employing the Banach fixed-point theorem in a special way. Known results are simple corollaries of his theorem.

Let the $x \equiv 0$ be eventually uniform-asymptotically stable (EvUAS) for the system $x' = f(t, x)$. A. STRAUSS gives sufficient conditions so that EvUAS is preserved under various types of perturbations. Numerous examples are given.

The Conference was attended by 163 registered participants. A total of 26 invited lecturers and 66 participants received stipends for travel and living expenses from a

grant by the Office of Naval Research. It is a pleasure to acknowledge this support with deep thanks; for not only did these funds make our Conference and these Proceedings possible, but above all they gave us an opportunity to bring together a large group of researchers in differential equations who had an opportunity to discuss problems of common interest for several days. It is such exchanges of ideas which are most rewarding and result in progress in research.

We are grateful to the invited lecturers for their mathematical contributions which resulted in an interesting and rewarding conference, and for their willingness to accept only travel and living expenses for their efforts. This generosity made it possible to support a large number of people on a relatively small budget. The University of Wisconsin made a significant contribution by giving us free use of Van Vleck Hall and other facilities. The Society for Industrial and Applied Mathematics deserves our thanks and appreciation, not only for sponsoring the Conference, but also for handling so efficiently the publication of these Proceedings. Finally, I acknowledge with pleasure and deep thanks the help of my colleagues, Fred Brauer, Charles Conley, Jacob Levin and David Russell, in organizing this conference, that of Richard Churchill for meticulous proof reading of this manuscript, and that of Mrs. Diane Rinehart Christensen who acted as secretary for the Conference and attended to the endless stream of details.

Madison, Wisconsin JOHN A. NOHEL
December 1968

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IN MEMORIAM

RUDOLPH E. LANGER
Professor of Mathematics
University of Wisconsin

March 8, 1894 to March 11, 1968

CHAPTER 1

Problems in Partial Differential Equations

Invited Lectures

Local Behavior of Solutions of Nonlinear Parabolic Equations

D. G. Aronson, *University of Minnesota*

Introduction. In 1956 John Nash [14] and E. de Giorgi [6] independently and by quite different methods derived a priori estimates for the Hölder continuity of weak solutions of divergence structure parabolic and elliptic differential equations. These estimates are independent of any smoothness assumptions on the equation and they settled one of the outstanding open questions in the theory of partial differential equations. Almost immediately after their publication, the Nash–de Giorgi methods and results were exploited by Ladyzhenskaya and Ural'tseva [10] and Oleinik and Kruzhkov [16] to establish the existence of solutions to the first boundary value problem for various nonlinear parabolic equations and systems. In 1959 J. Moser [11] found a new proof of de Giorgi's theorem. Moser's proof is considerably simpler than the original proof and his methods have provided some of the basic tools for very fruitful further investigations. Using essentially the methods of [11] together with the John–Nirenberg lemma and its variants Moser derived Harnack inequalities for the simplest linear divergence structure elliptic and parabolic equations [12], [13]. Serrin further refined Moser's techniques and applied them in his comprehensive study of the local behavior of solutions of quasilinear divergence structure elliptic equations [18], [19]. In the first part of this lecture we shall survey some recent work on quasilinear parabolic equations which grew out of Serrin's work on elliptic equations.

In all of the work described above it is, of course, necessary to assume something about the structure of the differential equation. More specifically, it is always required that equations in question be parabolic or elliptic (in some sense) for all admissible values of the variables. We shall call such equations *nondegenerate*. There are, however, several classes of physical problems which lead to degenerate partial differential equations, for example, the Prandtl boundary layer theory and the flow of a gas through a porous medium. While at least some parts of the theory of nondegenerate equations are reasonably complete, there are only very fragmentary results on nonlinear degenerate problems. In the second part of this lecture we shall survey some results for a specific degenerate problem, namely, the flow of a gas through a porous medium. Other degenerate problems are discussed in Oleinik's lecture [15].

Nondegenerate equations. Let $x = (x_1, \cdots, x_n)$ denote points in n-dimensional Euclidean space E^n, $n \geqq 1$, and let t denote points on the real line. Let Ω be a bounded domain in E^n and consider the cylinder $Q = \Omega \times (0, T]$ for some fixed $T > 0$. For $(x, t) \in Q$ we consider the second order quasilinear equation

$$u_t = \operatorname{div} A(x, t, u, u_x) + B(x, t, u, u_x), \tag{1}$$

where $A = (A_1, \cdots, A_n)$ is a given vector function of (x, t, u, u_x), B a given scalar function of the same variables, and $u_x = (\partial u/\partial x_1, \cdots, \partial u/\partial x_n)$. Also here div A refers to the divergence of the vector $A(x, t, u(x, t), u_x(x, t))$ with respect to the variables $(x_1, \cdots, x_n)$. The functions $A(x, t, u, p)$ and $B(x, t, u, p)$ are assumed to be defined and measurable for all $(x, t) \in Q$ and for all values u and p. Moreover, we suppose that A and B satisfy inequalities of the form

$$A(x, t, u, p) \cdot p \geqq a|p|^\alpha - b^\alpha|u|^\alpha - f^\alpha, \tag{2a}$$

$$|B(x, t, u, p)| \leqq c|p|^{\alpha-1} + d^{\alpha-1}|u|^{\alpha-1} + g^{\alpha-1}, \tag{2b}$$

$$|A(x, t, u, p)| \leqq \bar{a}|p|^{\alpha-1} + e^{\alpha-1}|u|^{\alpha-1} + h^{\alpha-1}. \tag{2c}$$

Here α is a constant $\geqq 1$, a and $\bar{a}$ are positive constants, while the coefficients $b, c, \cdots, h$ are, in general, non-negative functions of (x, t), each of which belongs to some appropriate Lebesgue class. The inequality (2a) can be taken as the definition of nondegeneracy for our present purpose.

Without further hypothesis on A and B it is not possible, in general, to speak of a classical solution of (1), and it is correspondingly necessary to introduce the notion of a generalized solution. To avoid unprofitable technical complications we shall refrain from doing this here; the term "solution of (1)" will mean either a weak solution as defined in [4] or, to those willing to make the necessary qualitative assumptions, a classical solution.

Let us assume first that $\alpha = 2$ and that each of the coefficients $b, c, \cdots, h$ in the inequalities (2a, b, c) belong to some space $L^{p,q}(Q) \equiv L^q(0, T; L^p(\Omega))$, where p and q are non-negative real numbers such that

$$p > 2 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < \frac{1}{2} \quad \text{for} \quad b, c, e, f, h, \tag{3a}$$

and

$$p > 1 \quad \text{and} \quad \frac{n}{2p} + \frac{1}{q} < 1 \quad \text{for} \quad d, g. \tag{3b}$$

The norms of the functions $b, c, \cdots, h$ in their respective spaces will be denoted simply by $\|b\|, \|c\|, \cdots, \|h\|$. Moreover, θ will denote a certain well-defined positive number determined by the exponents p, q occurring in the conditions (3a, b). It is easily verified that the linear equation

$$u_t = \{A_{ij}(x, t)u_{x_i} + A_j(x, t)u + F_j(x, t)\}_{x_j} + B_j(x, t)u_{x_j} + C(x, t)u + G(x, t) \tag{4}$$

satisfies the preceding hypotheses if: (i) the $A_{ij} \in L^\infty(Q)$ and there exists a constant $\nu > 0$ such that $A_{ij}(x, t)\xi_i\xi_j \geqq \nu|\xi|^2$ for almost all $(x, t) \in Q$ and all $\xi \in E^n$; (ii) the