# Natural Boundary Integral Method and Its Applications

Dehao Yu

SCIENCE PRESS
Beijing

(O–1204.0102)

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Boundary element methods are very important for solving boundary value problems in PDEs. Many boundary value problems of partial diffferential equations can be reduced into boundary integral equations by the natural boundary reduction. In this book the natural boundary integral method, suggested and developed by Feng and Yu, is introduced systematically. It is quite different from popular boundary element methods and has many distinctive advantages. The variational principle is conserved after the natural boundary reduction, and some useful properties are also preserved faithfully. Moreover, it can be applied directly and naturally in the coupling method and the domain decomposition method of finite and boundary elements. Most of the material in this book has only appeared in the author's previous papers. Compared with its Chinese edition (Science Press, Beijing, 1993), many new research results such as the domain decomposition methods based on the natural boundary reduction are added.

This book is intended for graduate students and researchers of computational and applied mathematics, scientific computing, computational mechanics and physics. It is also of interest to university lecturers, scientists and engineers who are interested in the application of the boundary element method.


ISBN 7–03–008270–2
定 价：128.00 元

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Published by Science Press
16 Donghuangchenggen North Street
Beijing 100717, China
Second Printed in 2006

Printed in Beijing



ISBN 7-03-008270-2 (Science Press, Beijing)

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*With deep respect to the memory of*

*Professor Kang FENG* (1920–1993)

# Preface

In the last twenty years the boundary integral methods, or the boundary element methods, as one kind of important numerical ways for solving partial differential equations, have been rapidly developed and widely applied in numerous fields of scientific and engineering computation. There are already a large number of papers on these methods and their applications. In this book the natural boundary integral method, or the natural boundary element method, is introduced. It has many distinctive advantages and is quite different from the popular classical boundary element methods. That method was first suggested and developed by Chinese mathematicians. As early as the late 1970s, Professor Kang FENG, who was my doctoral adviser, had already proposed the idea of the natural boundary reduction, which was called the canonical boundary reduction at that time. It was his initial work and his direct advice that brought me to this research field and encouraged me to develop this method systematically. Now the natural boundary integral method, which has attracted a good deal of attention and interest, is already a new branch of the boundary integral methods. It is only natural that emphasis should here be placed on the contribution of Professor Kang FENG to this method.

This book is intended for graduate students, students in the higher grades, university teachers and scientists, who study or are engaged in computational mathematics, computational mechanics, applied mathematics or other related special fields. Moreover, it is also a reference book for engineers who are interested in the application of boundary element methods. The book comprises seven chapters. Chapter I presents a general survey of the natural boundary integral method, which consists of the basic idea of the natural boundary reduction, the numerical

computation of hypersingular integrals, and the error estimates for the natural boundary integral method. Chapters II to V discuss the natural boundary reduction and the natural boundary integral method for the harmonic boundary value problem, the biharmonic boundary value problem, the plane elasticity problem and the Stokes problem, respectively, which contains the representation of a solution via complex variable functions, natural integral equations and Poisson integral formulas for some typical domains, the characters of natural integral operators, and the numerical solution of natural integral equations. Chapter VI expounds the coupling of natural boundary elements and finite elements, and the approximation of boundary conditions at infinity. Chapter VII introduces the domain decomposition methods based on the natural boundary reduction.

This book is a summary of my research results on the natural boundary integral method, which consists of a series of papers published in the last twenty years, and some results never published. Many contents of the book were also presented in my courses given at the Graduate School of the Chinese Academy of Sciences and several other universities since 1988. This book does not pursue perfection, therefore, many contents on the popular classical boundary element methods and their applications, which can be found in other books, are only sketched in §1.2 of Chapter I.

The Chinese edition of this book, with the title *"Mathematical Theory of the Natural Boundary Element Method"*, at first was published by Science Press in 1993 in Beijing. Since then eight years have already passed. In the spring of 1993, when I had just received some new cloth-bound copies of my book, Professor Kang FENG, my teacher and former supervisor, Member of Chinese Academy of Sciences, was still living and in good health. When I presented one copy of this book to him with my own hands, he was very happy and excited. This research direction, the natural boundary reduction and the natural boundary integral method, was opened up by himself. My book was just the development and implementation of his creative idea. For the sake of successful publication of this book, he and Professor Zhong-ci Shi, who is also one of my teachers and Member of the Chinese Academy of Sciences, wrote letters of recommendation to the committee of the Science Publication Foundation. They wrote: "This book is totally different from other existing books which deal with boundary element methods. It is an academic monograph at

an advanced international level, has many distinctive features, and is a reflection of research results of Chinese mathematicians in this direction. ··· The academic ideology of this book is novel. Its content consists of the author's research results in this direction over more than ten years. It is of great value to publish it". Thanks to their support, my book was published by Science Press, the most authoritative academic publishing house in China, as one of the series on monographs on pure mathematics and applied mathematics.

After my book had been published in Chinese, it was well received by its readers. Furthermore, it has also been followed with interest by some foreign mathematicians, even though they cannot understand the Chinese characters. Prof. W.L. Wendland from Germany, Prof. H. Fujita, Prof. T. Ushijima, Prof. K. Onishi from Japan, Dr. G.N. Makrakis from Greece, and some other foreign friends asked me for my book. There are some very nice words in the letter of October 11, 1996 from Prof. Fujita: "Although I need a little of my imagination to read mathematical papers and books written in Chinese, I can understand and appreciate your book. Indeed, your book contains basic as well as advanced facts and results concerning important topics from functional analysis and applied analysis. With my memory of Professor FENG Kang, your teacher and my friend, I have renewed my respect for the idea and methods of your school. This is my first impression on seeing your book.··· I am particularly pleased to find that you and I can share common interests in the application of the operator- theoretic approach to boundary value problems by means of your '$\mathcal{K}$' operator $\sim$ our '$S$' operator, which is a modern version of potential theory." Of course, most foreign friends cannot read my book written in Chinese, even if they use their imagination. I believe there is a need to translate this book from Chinese into English. In fact, since this book was published in Chinese, Mr. Peng Lin, the responsible editor of the book from Science Press, asked me many times for a translation of my book and has made efforts for the publication of its English version. However, because I have been immersed in other work, this English edition has been delayed until now.

Moreover, since 1993, the natural boundary integral method has also seen some new developments and applications. Its importance in theory, as well as in practice, is recognized by more and more scientists

who work in interrelated fields. Especially in recent years, with the developments of the coupling methods and the domain decomposition methods, which are the main trends in the research of the finite and boundary element methods today, the advantages of the natural boundary reduction become increasingly remarkable. Many theoretical analyses for the coupling methods are based on the natural boundary reduction. The natural boundary integral operators, *i.e.*, so called Dirichlet–Neumann operators, or Steklov–Poincaré operators, also play a key role in the domain decomposition methods.

To the creation, development and application of the natural boundary reduction and the natural boundary integral method, Professor Kang FENG made great contributions. We can find his creative idea in the keynote lecture of the China–France Symposium on the Finite Element Method (jointly with De-hao Yu) in 1982, and the invited lecture of the International Congress of Mathematicians in 1983. After ten years his academic interest was changed. He proposed and developed symplectic algorithms for solving evolution equations in Hamiltonian form. Even then he was still emphasizing the advantages of the natural boundary reduction time and again. In 1991 he said: "The natural boundary reduction is fully compatible with the variational principle over the domain, and the boundary finite elements are also fully compatible with the domain finite elements. So the resulting integrated methodology (domain finite elements plus boundary elements) is highly flexible and good for largescale problems with natural and compatible coupling between domains and boundaries. This is, in fact, a pioneering contribution to the so-called domain decomposition methods now under active developments in relation to parallel computation." (J. Comput. Math., 1991.) And then in 1993 he wrote: "The standard technology for solving boundary value problems is the finite element method. However, for complex problems involving infinite and/or cracked subdomains, re-entrant corners, intersecting interfaces, etc., the computing cost could be high. One may conceive an integrated FEM system with coupled BEM. There are many different ways of the boundary reduction. The best one seems to be the natural boundary reduction: to delete a troublesome subdomain by using a Green function of the first kind to get the exact artificial boundary condition with hypersingular integral of the Hadamard finite part. The real

merits are: first, the boundary reduction leaves the variational functional invariant, so the coupling between the boundary elements and the remaining well-behaved domain finite elements is direct and natural; second, the hyper-singularity actually improves stability, and effective quadratures are available. Natural boundary reduction can be directly used as a variant of domain decomposition plus deletion and indirectly applied to preconditioning problems." (Collected Works of FENG Kang, 1994.)

Under the guidance of Professor Feng's judgments, the author has suggested and developed some new domain decomposition methods which are based on the natural boundary reduction, quite different from standard methods, and especially suited to problems in unbounded domains. These new results are developments and applications of the theory of natural boundary integral operators, and are written in this book. Therefore in this English edition, a seventh chapter, *Domain decomposition methods based on natural boundary reduction*, is added, its content having been drawn from the author's recent papers. Consequently the title of this book has been changed to *The Natural boundary integral method and its applications*. Furthermore, in the second chapter, a subsection, *A theorem for general simply connected domains*, is added, its content being drawn from a joint paper with Kang FENG, which was published in 1994.

In this book we have only discussed 2-dimensional problems. Actually, it is more significant but more difficult to develop the natural boundary reduction for 3-dimensional cases. Besides, for Helmholtz, anisotropic and parabolic equations, for elliptic artificial boundaries, their natural boundary reductions and domain decomposition methods need also to be developed. However, these contents are still not discussed in this book, although we have already obtained many results in these directions, for which the readers can see our recent papers [123–126, 172, 198–201] and [203–206].

To our great sorrow, Professor Kang FENG died on August 17, 1993, at the age of 73 years. He was never able to read this English version, but I believe that if he were alive today, he would be very happy with its publication and for more and more developments and applications of the natural boundary reduction; all of these results are based on his creative idea. This English book is the best expression of my deep memory of him.

Mr. Peng Lin, the director of the scientific publishing center, Science Press, payed very close attention to the English edition of this book from beginning to end. Mr. Jia-shan Liu from Science Press, Mr. Xiang-yang Ge from the Institute of Computational Mathematics, and editors from Kluwer Academic Publishers all put in a lot of hard work for its publication. Allow me to express deeply my thanks to them. I would also like to thank my son, Xin-wei Yu, a Ph.D. student of the California Institute of Technology, for his typing of all mathematical formulas of this book in LaTeXwhen he was studying at the Peking University in Beijing.

De-hao Yu
March, 2001
Beijing

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