LINEAR PROGRAMMING AND NETWORK FLOWS

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PREFACE

Linear Programming deals with the problem of minimizing or maximizing a linear function in the presence of linear inequalities. Since the development of the simplex method by George B. Dantzig in 1947, linear programming has been extensively used in the military, industrial, governmental, and urban planning fields, among others. The popularity of linear programming can be attributed to many factors including its ability to model large and complex problems, and the ability of the users to solve large problems in a reasonable amount of time by the use of the simplex method and computers.

During and after World War II it became evident that planning and coordination among various projects and the efficient utilization of scarce resources were essential. Intensive work by the U. S. Air Force team SCOOP (Scientific Computation of Optimum Programs) began in June 1947. As a result, the simplex method was developed by George B. Dantzig by the end of summer 1947. Interest in linear programming spread quickly among economists, mathematicians, statisticians, and government institutions. In the summer of 1949 a conference on linear programming was held under the sponsorship of the Cowles Commission for Research in Economics. The papers presented at that conference were later collected in 1951 by T. C. Koopmans into the book *Activity Analysis of Production and Allocation.*

Since the development of the simplex method many people have contributed to the growth of linear programming by developing its mathematical theory, devising efficient computational methods and codes, exploring new applications, and by their use of linear programming as an aiding tool for solving more complex problems, for instance, discrete programs, nonlinear programs, combinatorial problems, stochastic programming problems, and problems of optimal control.

This book addresses the subjects of linear programming and network flows. The simplex method represents the backbone of most of the techniques presented in the book. Whenever possible, the simplex method is specialized to take advantage of problem structure. Throughout we have attempted first to present the techniques, to illustrate them by numerical examples, and then to provide detailed mathematical analysis and an argument showing convergence to an optimal solution. Rigorous proofs of the results are given without the theorem-proof format. Even though this may bother some readers, we believe that the format and mathematical level adopted in this book will provide an adequate and smooth study for those who wish to learn the techniques and the know-how to use them, and for those who wish to study the algorithms at a more rigorous level.

The book can be used both as a reference and as a textbook for advanced undergraduate students and first-year graduate students in the fields of industrial engineering, management, operations research, computer science, mathematics, and other engineering disciplines that deal with the subjects of linear programming and network flows. Even though the book's material requires some mathematical maturity, the only prerequisite is linear algebra. For convenience of the reader, pertinent results from linear algebra and convex analysis are summarized in Chapter two. In a few places in the book, the notion of differentiation would be helpful. These, however, can be omitted without loss of understanding or continuity.

This book can be used in several ways. It can be used in a two-course sequence on linear programming and network flows, in which case all of its material could be easily covered. The book can also be utilized in a one-semester course on linear programming and network flows. The instructor may have to omit some topics at his discretion. The book can also be used as a text for a course on either linear programming or network flows.

Following the introductory first chapter and the second chapter on linear algebra and convex analysis, the book is organized into two parts: linear programming and networks flows. The linear programming part consists of Chapters three to seven. In Chapter three the simplex method is developed in detail, and in Chapter four the initiation of the simplex method by the use of artificial variables and the problem of degeneracy are discussed. Chapter five deals with some specializations of the simplex method and the development of optimality criteria in linear programming. In Chapter six we consider the dual problem, develop several computational procedures based on duality, and discuss sensitivity and parametric analysis. Chapter seven introduces the reader to the decomposition principle and to large-scale programming. The part on network flows consists of Chapters eight to eleven. Many of the procedures in this part are presented as a direct simplification of the simplex method. In Chapter eight the transportation problem and the assignment problem are both examined. Chapter nine considers the minimal cost network flow problem from the simplex method point of view. In Chapter ten we present the out-of-kilter algorithm for solving the same problem. Finally, Chapter eleven covers the special topics of the maximal flow problem, the shortest path problem, and the multicommodity minimal cost flow problem.

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ONE: INTRODUCTION

In 1949 George B. Dantzig published the "simplex method" for solving linear programs. Since that time a number of individuals have contributed to the field of linear programming in many different ways including theoretical development, computational aspects, and exploration of new applications of the subject. The simplex method of linear programming enjoys wide acceptance because of (1) its ability to model important and complex management decision problems and (2) its capability for producing solutions in a reasonable amount of time. In subsequent chapters of this text we shall consider the simplex method and its variants, with emphasis on the understanding of the methods.

In this chapter the linear programming problem is introduced. The following topics are discussed: basic definitions in linear programming, assumptions leading to linear models, manipulation of the problem, examples of linear problems, and geometric solution in the feasible region space and the requirement space. This chapter is elementary and may be skipped if the reader has previous knowledge of linear programming.

1.1 THE LINEAR PROGRAMMING PROBLEM

A linear programming problem is a problem of minimizing or maximizing a linear function in the presence of linear constraints of the inequality and/or the equality type. In this section the linear programming problem is formulated.

Basic Definitions

Consider the following linear programming problem.

$$
\begin{array}{llllll}
\text{Minimize} & c_1x_1 + & c_2x_2 + & \cdots + & c_nx_n & \\
\text{Subject to} & a_{11}x_1 + & a_{12}x_2 + & \cdots + & a_{1n}x_n \geqslant & b_1 \\
 & a_{21}x_1 + & a_{22}x_2 + & \cdots + & a_{2n}x_n \geqslant & b_2 \\
 & \vdots & \vdots & & \vdots & \vdots \\
 & a_{m1}x_1 + & a_{m2}x_2 + & \cdots + & a_{mn}x_n \geqslant & b_m \\
 & x_1, & x_2, & \ldots, & x_n \geqslant & 0
\end{array}
$$

Here $c_1x_1 + c_2x_2 + , \ldots, + c_nx_n$ is the *objective function* (or criterion function) to be minimized and will be denoted by z. The coefficients $c_1, c_2, \ldots, c_n$ are the (known) *cost coefficients* and $x_1, x_2, \ldots, x_n$ are the *decision variables* (variables, or activity levels) to be determined. The inequality $\sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$ denotes the ith *constraint* (or restriction). The coefficients a_{ij} for $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$ are called the *technological coefficients*. These technological coefficients form the *constraint matrix* **A** given below.

$$
\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}
$$

The column vector whose ith component is b_i, which is referred to as the *right-hand-side vector*, represents the minimal requirements to be satisfied. The constraints $x_1, x_2, \ldots, x_n \geqslant 0$ are the *nonnegativity constraints*. A set of variables $x_1, \ldots, x_n$ satisfying all the constraints is called a *feasible point* or a *feasible vector*. The set of all such points constitutes the *feasible region* or the *feasible space*.

Using the foregoing terminology, the linear programming problem can be stated as follows: Among all feasible vectors, find that which minimizes (or maximizes) the objective function.

Example 1.1

Consider the following linear problem.

$$\text{Minimize } 2x_1 + 5x_2$$

$$\begin{aligned} \text{Subject to } \quad x_1 + x_2 &\geq 6 \\ -x_1 - 2x_2 &\geq -18 \\ x_1, \ x_2 &\geq 0 \end{aligned}$$

In this case we have two decision variables x_1 and x_2. The objective function to be minimized is $2x_1 + 5x_2$. The constraints and the feasible region are illustrated in Figure 1.1. The optimization problem is thus to find a point in the feasible region with the smallest possible objective.

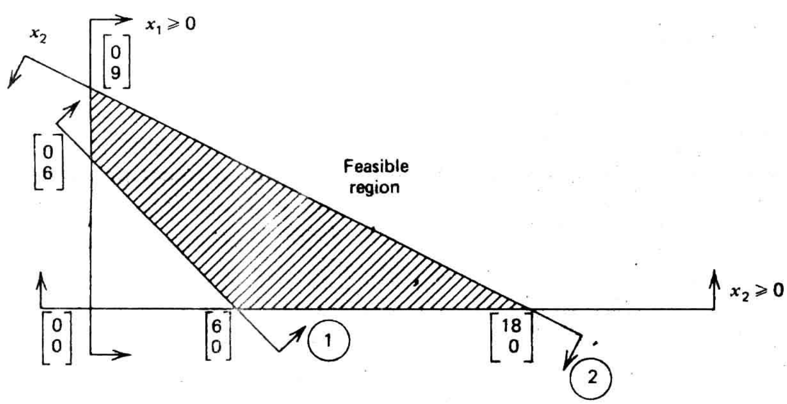

Figure 1.1. Illustration of the feasible region.

Assumptions of Linear Programming

In order to represent an optimization problem as a linear program, several assumptions that are implicit in the linear programming formulation discussed above are needed. A brief discussion of these assumptions is given below.

1. *Proportionality.* Given a variable x_j, its contribution to cost is c_jx_j and its contribution to the ith constraint is $a_{ij}x_j$. This means that if x_j is doubled, say, so is its contribution to cost and to each of the constraints. To illustrate, suppose that x_j is the amount of activity j used. For instance, if

$x_j = 10$, then the cost of this activity is $10c_j$. If $x_j = 20$, then the cost is $20c_j$, and so on. This means that no savings (or extra costs) are realized by using more of activity j. Also no setup cost for starting the activity is realized.

2. *Additivity*. This assumption guarantees that the total cost is the sum of the individual costs, and that the total contribution to the ith restriction is the sum of the individual contributions of the individual activities.
3. *Divisibility*. This assumption ensures that the decision variables can be divided into any fractional levels so that noninteger values for the decision variables are permitted.

To summarize, an optimization problem can be cast as a linear program only if the aforementioned assumptions hold. This precludes situations where economies of scale exist; for example, when the unit cost decreases as the amount produced is increased. In these situations one must resort to nonlinear programs. It should also be noted that the parameters c_j, a_{ij}, and b_i must be known or estimated.

Problem Manipulation

Recall that a linear program is a problem of minimizing or maximizing a linear function in the presence of linear inequality and/or equality constraints. By simple manipulations the problem can be transformed from one form to another equivalent form. These manipulations are most useful in linear programming, as will be seen throughout the text.

INEQUALITIES AND EQUATIONS

An inequality can be easily transformed into an equation. To illustrate, consider the constraint given by $\sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$. This constraint can be put in an equation form by subtracting the nonnegative *slack variable* x_{n+i} (sometimes denoted by s_i) leading to $\sum_{j=1}^{n} a_{ij}x_j - x_{n+i} = b_i$ and $x_{n+i} \geqslant 0$. Similarly the constraint $\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i$ is equivalent to $\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i$ and $x_{n+i} \geqslant 0$. Also an equation of the form $\sum_{j=1}^{n} a_{ij}x_j = b_i$ can be transformed into the two inequalities $\sum_{j=1}^{n} a_{ij}x_j \leqslant b_i$ and $\sum_{j=1}^{n} a_{ij}x_j \geqslant b_i$.

NONNEGATIVITY OF THE VARIABLES

For most practical problems the variables represent physical quantities and hence must be nonnegative. The simplex method is designed to solve linear programs where the variables are nonnegative. If a variable x_j is unrestricted in sign, then it can be replaced by $x_j' - x_j''$ where $x_j' \geqslant 0$ and $x_j'' \geqslant 0$. If $x_j \geqslant l_j$, then the new variable $x_j' = x_j - l_j$ is automatically nonnegative. Also if a variable x_j is restricted such that $x_j \leqslant u_j$ where $u_j \leqslant 0$, then the substitution $x_j' = u_j - x_j$ produces a nonnegative variable x_j'.

MINIMIZATION AND MAXIMIZATION PROBLEMS

Another problem manipulation is to convert a maximization problem into a minimization problem and conversely. Note that over any region

$$\text{Maximum} \sum_{j=1}^{n} c_j x_j = -\text{minimum} \sum_{j=1}^{n} -c_j x_j$$

So a maximization (minimization) problem can be converted into a minimization (maximization) problem by multiplying the coefficients of the objective function by -1. After the optimization of the new problem is completed, the objective of the old problem is -1 times the optimal objective of the new problem.

Standard and Canonical Formats

From the foregoing discussion we see that a given linear program can be put in different equivalent forms by suitable manipulations. Two forms in particular will be useful. These are the standard and the canonical forms. A linear program is said to be in *standard format* if all restrictions are equalities and all variables are nonnegative. The simplex method is designed to be applied only after the problem is put in the standard form. The canonical form is also useful especially in exploiting duality relationships. A minimization problem is in *canonical form* if all variables are nonnegative and all the constraints are of the $\geqslant$ type. A maximization problem is in canonical format if all the variables are nonnegative and all the constraints are of the $\leqslant$ type. The standard and canonical forms are summarized in Table 1.1.

Linear Programming in Matrix Notation

A linear programming problem can be stated in a more convenient form using matrix notation. To illustrate, consider the following problem.

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, 2, \ldots, m$$

$$x_j \geqslant 0 \quad j = 1, 2, \ldots, n$$

Table 1.1 Standard and Canonical Forms

	MINIMIZATION PROBLEM	MAXIMIZATION PROBLEM
Standard Form	Minimize $\sum_{j=1}^{n} c_j x_j$	Maximize $\sum_{j=1}^{n} c_j x_j$
	Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$	Subject to $\sum_{j=1}^{n} a_{ij} x_j = b_i \quad i = 1, \ldots, m$
	$x_j \geqslant 0 \quad j = 1, \ldots, n$	$x_j \geqslant 0 \quad j = 1, \ldots, n$
Canonical Form	Minimize $\sum_{j=1}^{n} c_j x_j$	Maximize $\sum_{j=1}^{n} c_j x_j$
	Subject to $\sum_{j=1}^{n} a_{ij} x_j \geqslant b_i \quad i = 1, \ldots, m$	Subject to $\sum_{j=1}^{n} a_{ij} x_j \leqslant b_i \quad i = 1, \ldots, m$
	$x_j \geqslant 0 \quad j = 1, \ldots, n$	$x_j \geqslant 0 \quad j = 1, \ldots, n$

Denote the row vector $(c_1, c_2, \ldots, c_n)$ by $\mathbf{c}$, and consider the following column vectors $\mathbf{x}$ and $\mathbf{b}$, and the $m \times n$ matrix $\mathbf{A}$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Then the above problem can be written as follows.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{cx} \\ \text{Subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

The problem can also be conveniently represented via the columns of $\mathbf{A}$. Denoting $\mathbf{A}$ by $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n]$ where $\mathbf{a}_j$ is the jth column of $\mathbf{A}$, the problem can be formulated as follows.

$$\begin{aligned} \text{Minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{Subject to} \quad & \sum_{j=1}^{n} \mathbf{a}_j x_j = \mathbf{b} \\ & x_j \geqslant 0 \qquad j = 1, 2, \ldots, n \end{aligned}$$

1.2 EXAMPLES OF LINEAR PROBLEMS

In this section we describe several problems that can be formulated as linear programs. The purpose is to show the varieties of problems that can be recognized and expressed in precise mathematical terms as linear programs.

Feed Mix Problem

An agricultural mill manufactures feed for chickens. This is done by mixing several ingredients, such as corn, limestone, or alfalfa. The mixing is to be done in such a way that the feed meets certain levels for different types of nutrients, such as protein, calcium, carbohydrates, and vitamins. To be more specific, suppose that n ingredients $j = 1, 2, \ldots, n$ and m nutrients $i = 1, 2, \ldots, m$ are considered. Let the unit cost of ingredient j be c_j and let the amount of ingredient j to be used be x_j. The cost is therefore $\sum_{j=1}^{n} c_j x_j$. If the amount of the final product needed is b, then we must have $\sum_{j=1}^{n} x_j = b$. Further suppose that a_{ij} is the amount of nutrient i present in a unit of ingredient j, and that the acceptable lower and upper limits of nutrient i in a unit of the chicken feed are l_i' and u_i' respectively. Therefore we must have the constraints $l_i'b \leq \sum_{j=1}^{n} a_{ij} x_j \leq u_i'b$ for $i = 1, 2, \ldots, m$. Finally, because of shortages, suppose that the mill cannot acquire more than u_j units of ingredient j. The problem of mixing the ingredients such that the cost is minimized and the above restrictions are met, can be formulated as follows.

$$
\begin{array}{llcl}
\text{Minimize} & & c_1x_1 + c_2x_2 \cdots + c_nx_n & \\
\text{Subject to} & & x_1 + x_2 \cdots + x_n = b & \\
& bl_1' \leq & a_{11}x_1 + a_{12}x_2 \cdots + a_{1n}x_n & \leq bu_1' \\
& bl_2' \leq & a_{21}x_1 + a_{22}x_2 \cdots + a_{2n}x_n & \leq bu_2' \\
& \vdots & \vdots \qquad \vdots \qquad\qquad \vdots & \vdots \\
& bl_m' \leq & a_{m1}x_1 + a_{m2}x_2 \cdots + a_{mn}x_n & \leq bu_m' \\
& & 0 \leq x_1 \leq u_1 & \\
& & 0 \leq x_2 \leq u_2 & \\
& & \vdots & \\
& & 0 \leq x_n \leq u_n &
\end{array}
$$

Production Scheduling: An Optimal-Control Problem

A company wishes to determine the production rate over the planning horizon of the next T weeks such that the known demand is satisfied and the total production and inventory cost is minimized. Let the known demand rate at time t be $g(t)$, and similarly denote the production rate and inventory at t by $x(t)$ and $y(t)$. Further suppose that the initial inventory at time 0 is y_0 and that the desired inventory at the end of the planning horizon is y_T. Suppose that the inventory cost is proportional to the units in storage, so that the inventory cost is given by $c_1 \int_0^T y(t)\,dt$ where $c_1 > 0$ is known. Also suppose that the production cost is proportional to the rate of production, and so is given by $c_2 \int_0^T x(t)\,dt$. Then the total cost is $\int_0^T [c_1 y(t) + c_2 x(t)]\,dt$. Also note that the inventory at any time is given according to the relationship

$$y(t) = y_0 + \int_0^t \left[x(\tau) - g(\tau)\right] d\tau \qquad t \in [0, T]$$