CISM COURSES AND LECTURES NO. 172
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

FORCED LINEAR VIBRATIONS

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PRINGER - VERLAG
IEN NEW YORK

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SPRINGER - VERLAG WIEN - NEW YORK

Printed in Italy

ISBN 3-211-81487-6 Springer-Verlag Wien-New York
ISBN 0-387-81487-6 Springer-Verlag New York-Wien

PREFACE

This textbook contains, with some extensions, our lectures given at the Department of General Mechanics of the International Centre for Mechanical Sciences (CISM) in Udine/Italy during the month of October, 1973.

The book is divided into four major parts. The first part (Chapter 2, 3) is concerned with the mathematical representation of vibration systems and the corresponding general solution. The second part (Chapter 4) deals with the boundedness and stability of vibration systems. Thus, information on the general behavior of the system is obtained without any specified knowledge of the initial conditions and forcing functions. The third part (Chapter 5, 6) is devoted to deterministic excitation forces. In particular, the harmonic excitation leads to the phenomena of resonance, pseudoresonance and absorption. The fourth part (Chapter 7) considers stochastic excitation forces. The covariance analysis and the spectral density analysis of random vibrations are presented. Throughout the book examples are inserted for illustration.

In conclusion, we wish to express our gratitude to the International Centre for Mechanical Sciences (CISM) and to Professor Sobrero who invited us to deliver the lecture in Udine. We also acknowledge the support of Professor Magnus from the Institute B of Mechanics at the Technical University Munich.

Munich, October 1973

Peter C. Müller *Werner O. Schiehlen*

CHAPTER 1

Introduction

The subject of vibration deals with the oscillatory behavior of physical systems. The interaction of mass and elasticity allows vibration as well as the interaction of induction and capacity. Most vehicles, machines and circuits experience vibration and their design generally requires consideration of their oscillatory behavior.

Vibration systems can be characterized as linear or non-linear, as time-invariant or time-variant, as free or forced, as single-degree of freedom or multi-degree of freedom. For linear systems the principle of superposition holds, and the mathematical techniques available for their treatment are well-developed in matrix and control theory. In contrast, for the analysis of nonlinear systems the techniques are only partially developed and they are based mainly on approximation methods. For linear, time-invariant systems the concept of modal analysis is available featuring eigenvalues and eigenvectors. In contrary, for the analysis of linear, time-variant systems the fundamental matrix has to be found by numerical integration. Free vibrations take place when a system oscillates without external impressed forces. The system under free vibration will oscillate at its natural frequencies or eigenfrequencies. In contrast, forced vibrations take place under the excitation of external forces, in particular, impulse, periodic and stochastic forces. Single-degree of freedom systems are characterized by a scalar differential equation of second order. In contrary, multi-degree of freedom systems are usually described by vector and matrix differential equations. The number of degrees of freedom is equal to the minimum number of generalized coordinates necessary to describe the motion of the system. In addition to the notions presented above, Magnus (1969) uses the notions self-excited and parameter-excited. Self-excited vibrations may occur in nonlinear time-invariant, free systems while parameter-excited vibrations are typical for linear, periodic time-invariant free systems.

In this contribution, linear, time-invariant forced vibrations of mechanical systems with multi-degrees of freedom will be considered. Linear time-invariant systems are often obtained by the linearization of mechanical systems in the neighborhood of an equilibrium position. Forced systems result in addition to free systems in many vital phenomena such as resonance, pseudo-resonance, absorption and random vibrations. Multi-degree of freedom systems are usually necessary for an adequate representation of mechanical systems. Even if a continuous elastic body has an infinite number of degrees of freedom, in many cases, part of such bodies may be assumed to be rigid and the system may be dynamically equivalent to one with finite degrees of freedom.

A rigorous treatment is given to the boundedness and stability of the system's vibration, to resonances including pseudo-resonance and absorption, and to the random vibration analysis via the covariance and the spectral density technique. The computer-minded matrix theory is applied and approved numerical algorithms are mentioned to serve the special needs of multi-degree of freedom systems. But simple examples are also analytically treated to achieve a better understanding.

CHAPTER 2

Mathematical Representation of Mechanical Vibration Systems

The mathematical representation of a mechanical system requires firstly an adequate model. Secondly, one of the principles of dynamics has to be applied to the model and, then, the equations of motion are obtained. Finally, the equations of motion can be summarized to the state equation of the vibration system.

2.1 Modeling of Vibration Systems

For the modeling of vibration systems four approaches can be listed:

1. Multi-body approach,
2. Finite element approach,
3. Continuous system approach,
4. Hybrid approach.

For each engineering problem, the appropriate approach has to be elected. Some examples may illustrate the proceeding.

The vibrations of an automobile suspension can be properly modeled by a three-body system, Fig. 2.1, where the automobile body and the wheels and axles are considered as rigid bodies connected by springs and dashpots.

Further, the elasticity of the tires is represented by springs without damping.

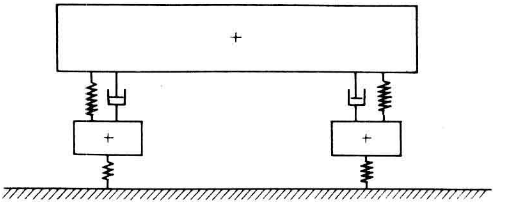

Fig.2.1. Three-body model of an automobile suspension

The vibrations of a spinning centrifuge with respect to its flexible suspension can be modeled by a rotating rigid body in the best manner, Fig. 2.2. The suspension is represented by spring and dashpot.

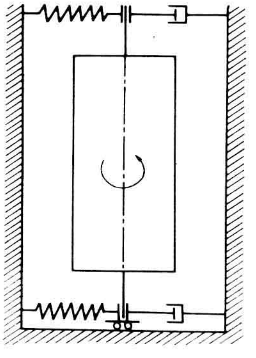

Fig.2.2. One-body model of a centrifuge

The bending vibrations of an automobile body have to be modeled by bar, rectangular and triangular elements, Fig. 2.3. Each element is considered as a flexible body where stiffness, damping and mass are concentrated in the nodes connecting the elements.

The torsional vibrations of a uniform bar are modeled best by a continuous system, Fig. 2.4. The infinite small elements are furnished with mass and elasticity.

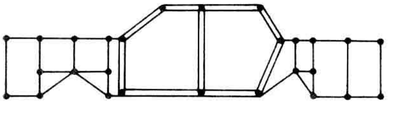

Fig.2.3. Finite element model of an automobile body

However, sometimes the three fundamental approaches do not fit the engineering problem as well. As an example the spinning flexible satellite may be mentioned. Here, the core body is surely a rigid body while the flexible appendages represent continuous bars. In such cases, the continuous system may be replaced by a large number of elastically interconnected rigid bodies.

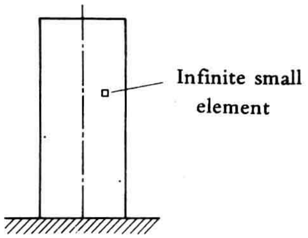

Fig.2.4. Continuous system model of a bar

Then, the multi-body approach can be used again. Or a hybrid approach, Fig. 2.5, is used combining the multi-body and the continuous system approach; see Likins (1971).

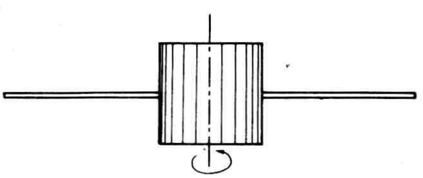

Fig.2.5. Hybrid model of a spinning satellite with flexible appendages

In the next sections the three fundamental approaches will be reviewed in short and the corresponding principles of dynamics will be applied.

2.2 Multi-body Approach

Assume a discrete, mechanical system with the following elements: rigid bodies with constraints, springs dashpots and actuators, Fig. 2.6. Then, either Euler's equation together with Newton's equation or Lagrange's equation may be applied. Both methods require the same kinematics.

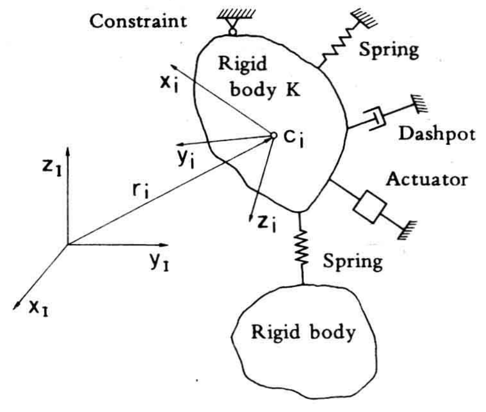

Fig.2.6. Discrete mechanical system with rigid bodies

Kinematics

The position of the rigid body K_i is uniquely characterized in space by a body-fixed, orthogonal frame. With respect to the inertial frame x_I, y_I, z_I, the body-fixed frame x_i, y_i, z_i, with origin at the center of mass C_i can be described by the 3×1-position vector $\mathbf{r}_i$ and the 3×3-rotation matrix $\mathbf{A}_i$. If there is only one free rigid body, then the position vector may be given by three Cartesian coordinates

$$\mathbf{r}_i = \begin{bmatrix} r_x & r_y & r_z \end{bmatrix}^T , \quad i = 1 , \tag{2.1}$$

and the rotation matrix may be given by three Euler angles, representing three generalized coordinates,

$$\mathbf{A}_i = \begin{bmatrix} \cos\theta \cos\psi & -\cos\theta \sin\psi & \sin\theta \\ \cos\phi \sin\psi + \sin\phi \sin\theta \cos\psi & \cos\phi \cos\psi - \sin\phi \sin\theta \sin\psi & -\sin\phi \cos\theta \\ \sin\phi \sin\psi - \cos\phi \sin\theta \cos\psi & \sin\phi \cos\psi + \cos\phi \sin\theta \sin\psi & \cos\phi \cos\theta \end{bmatrix}, \quad i = 1. \tag{2.2}$$

Obviously, the free rigid body has six degrees of freedom. However, if there is a system of p rigid bodies, possibly with some holonomic constraints, then the position vector and the rotation matrix of the body K_i may depend on all generalized coordinates of (translational) position as well as of rotation

$$(2.3) \qquad \left.\begin{array}{l} \mathbf{r}_i = \mathbf{r}_i(\mathbf{y}, t), \\ \mathbf{A}_i = \mathbf{A}_i(\mathbf{y}, t), \end{array}\right\} i = 1(1)p$$

where $\mathbf{y}$ is the $f \times 1$ -generalized position vector composed of the generalized coordinates. For the system's number of degrees of freedom it yields

$$(2.4) \qquad f \leqslant 6p .$$

The 3×1 -velocity vector $\mathbf{v}_i$ and the 3×1 -angular velocity vector $\boldsymbol{\omega}_i$ of the body K_i with respect to the inertial frame are obtained by differentiation of (2.3)

$$(2.5) \qquad \left.\begin{array}{ll} \mathbf{v}_i = \mathfrak{J}_{Ti}\,\dot{\mathbf{y}} + \bar{\mathbf{v}}_i, & \bar{\mathbf{v}}_i = \partial \mathbf{r}_i / \partial t, \\ \boldsymbol{\omega}_i = \mathfrak{J}_{Ri}\,\mathbf{y} + \bar{\boldsymbol{\omega}}_i, & \bar{\boldsymbol{\omega}}_i = \partial \mathbf{a}_i / \partial t, \end{array}\right\} i = 1(1)p$$

where

$$(2.6) \qquad \mathfrak{J}_{Ti} = \frac{\partial \mathbf{r}_i}{\partial \mathbf{y}} = \begin{bmatrix} \dfrac{\partial r_{xi}}{\partial y_1} & \dfrac{\partial r_{xi}}{\partial y_2} & \cdots & \dfrac{\partial r_{xi}}{\partial y_f} \\ \dfrac{\partial r_{yi}}{\partial y_1} & \dfrac{\partial r_{yi}}{\partial y_2} & \cdots & \dfrac{\partial r_{yi}}{\partial y_f} \\ \dfrac{\partial r_{zi}}{\partial y_1} & \dfrac{\partial r_{zi}}{\partial y_2} & \cdots & \dfrac{\partial r_{zi}}{\partial y_f} \end{bmatrix}, \quad i = 1(1)p$$

is the $3 \times f$ -Jacobian matrix of translation and

$$\mathfrak{J}_{Ri} = \frac{\partial \mathbf{a}_i}{\partial \mathbf{y}} = \begin{bmatrix} \frac{\partial a_{xi}}{\partial y_1} & \frac{\partial a_{xi}}{\partial y_2} & \cdots\cdots & \frac{\partial a_{xi}}{\partial y_f} \\ \frac{\partial a_{yi}}{\partial y_1} & \frac{\partial a_{yi}}{\partial y_2} & \cdots\cdots & \frac{\partial a_{yi}}{\partial y_f} \\ \frac{\partial a_{zi}}{\partial y_1} & \frac{\partial a_{zi}}{\partial y_2} & \cdots\cdots & \frac{\partial a_{zi}}{\partial y_f} \end{bmatrix}, \quad i = 1(1)p \qquad (2.7)$$

is the $3 \times f$ -Jacobian matrix of rotation. The angular velocity $\bar{\boldsymbol{\omega}}_i$ and the rotational Jacobian matrix $\mathfrak{J}_{Ri}$ are obtained from the corresponding skew-symmetric rotation tensors

$$\frac{\partial \tilde{\mathbf{a}}_i}{\partial t} = \frac{\partial \mathbf{A}_i}{\partial t} \cdot \mathbf{A}_i^T \ , \quad \frac{\partial \tilde{\mathbf{a}}_i}{\partial \mathbf{y}_j} = \frac{\partial \mathbf{A}_i}{\partial \mathbf{y}_j} \mathbf{A}_i^T \ , \quad \begin{matrix} i = 1(1)p \\ j = 1(1)f \end{matrix} \qquad (2.8)$$

where

$$\tilde{\mathbf{a}}_i = \begin{bmatrix} 0 & -a_{zi} & a_{yi} \\ a_{zi} & 0 & -a_{xi} \\ -a_{yi} & a_{xi} & 0 \end{bmatrix} \quad \text{for} \quad \mathbf{a}_i = \begin{bmatrix} a_{xi} \\ a_{yi} \\ a_{zi} \end{bmatrix} \qquad (2.9)$$

and $\mathbf{a}_i$ is a 3×1 -vector. Thus, $\sim$ characterizes the matrix notation of the vector cross product.

Newton's and Euler's Equation

Newton's equation reads for each rigid body K_i with respect to the center of mass C_i as

$$m_i \dot{\mathbf{v}}_i = \mathbf{f}_i \ , \quad i = 1(1)p \qquad (2.10)$$

where m_i is the scalar mass and $\mathbf{f}_i$ is the 3×1 -force vector including all forces acting on body K_i. Euler's equation reads for each body K_i with respect to C_i as

(2.11) $$\mathbf{I}_i \dot{\boldsymbol{\omega}}_i + \tilde{\boldsymbol{\omega}}_i \mathbf{I}_i \boldsymbol{\omega}_i = \mathbf{l}_i \, , \quad i = 1(1)p$$

where $\mathbf{I}_i$ is the 3×3 inertia tensor of body K_i and $\mathbf{l}_i$ is the 3×1 -torque vector including all torques acting on body K_i. The force $\mathbf{f}_i$ and the torque $\mathbf{l}_i$ depend in forced vibration systems on the generalized coordinates (spring forces), on the generalized velocities (dashpot forces), on the time (actuator forces) and on the constraints

(2.12) $$\left.\begin{aligned} \mathbf{f}_i &= \mathbf{f}_{Bi}(\mathbf{y}, \dot{\mathbf{y}}, t) + \mathbf{f}_{Ci} \, , \\ \mathbf{l}_i &= \mathbf{l}_{Bi}(\mathbf{y}, \dot{\mathbf{y}}, t) + \mathbf{l}_{Ci} \, , \end{aligned}\right\} \quad i = 1(1)p$$

where $\mathbf{f}_{Ci}$, $\mathbf{l}_{Ci}$ are due to the constraints.
Introducing (2.5) and (2.12) in (2.10),(2.11) it remains

(2.13) $$\begin{cases} m_i \mathbf{J}_{Ti} \ddot{\mathbf{y}} + m_i \dot{\mathbf{J}}_{Ti} \dot{\mathbf{y}} + m_i \dot{\bar{\mathbf{v}}}_i = \mathbf{f}_{Bi}(\mathbf{y}, \dot{\mathbf{y}}, t) + \mathbf{f}_{Ci} \, , \\ \mathbf{I}_i \mathbf{J}_{Ri} \ddot{\mathbf{y}} + \mathbf{I}_i \dot{\mathbf{J}}_{Ri} \dot{\mathbf{y}} + \mathbf{I}_i \dot{\bar{\boldsymbol{\omega}}}_i + \left(\widetilde{\mathbf{J}_{Ri} \dot{\mathbf{y}} + \bar{\boldsymbol{\omega}}_i}\right) \mathbf{I}_i \left(\mathbf{J}_{Ri} \dot{\mathbf{y}} + \bar{\boldsymbol{\omega}}_i\right) = \\ = \mathbf{l}_{Bi}(\mathbf{y}, \dot{\mathbf{y}}, t) + \mathbf{l}_{Ci} \, , \\ i = 1(1)p \end{cases}$$

The $6p$ scalar equations (2.13) can be summarized in matrix notation

(2.14) $$\bar{\mathbf{M}}(\mathbf{y}, t)\, \ddot{\mathbf{y}} + \bar{\mathbf{g}}(\dot{\mathbf{y}}, \mathbf{y}, t) + \bar{\mathbf{f}}_c = 0$$

where $\bar{\mathbf{M}}$ is a $6p \times f$ -mass matrix, $\bar{\mathbf{g}}$ is a $6p \times 1$ -vector function including $\bar{\mathbf{v}}_i$ and $\bar{\boldsymbol{\omega}}_i$ and $\mathbf{f}_c$ is the $6p \times 1$ -vector of the constraint forces and torques. Thus, one gets $6p$ equations for the f generalized coordinates and $6p-f$ linear independent constraint forces. Usually, however, the constraint forces are not required and for system order reduction they have, then, to be eliminated. This can be done by the principle of virtual work regarding (2.6), (2.7):

$$\sum_{i=1}^{p}\left(\mathbf{f}_{ci}^T\,\delta\mathbf{r}_i + \mathbf{l}_{ci}^T\,\delta\mathbf{a}_i\right) = \delta\mathbf{y}^T\sum_{i=1}^{p}\left(\mathfrak{J}_{Ti}^T\,\mathbf{f}_{ci} + \mathfrak{J}_{Ri}^T\,\mathbf{l}_{ci}\right) = 0 \qquad (2.15)$$

or

$$\bar{\mathfrak{J}}^T\,\bar{\mathbf{f}}_c = 0 \qquad (2.16)$$

where $\bar{\mathfrak{J}}^T = \left[\,\bar{\mathfrak{J}}_{T_1}^T\ \bar{\mathfrak{J}}_{T_2}^T\ \cdots\cdots\ \bar{\mathfrak{J}}_{Rp-1}^T\ \bar{\mathfrak{J}}_{Rp}^T\,\right]$ is the global $f \times 6p$-Jacobian matrix. Then, premultiplying (2.14) by $\bar{\mathfrak{J}}^T$, it remains

$$\mathbf{M}(\mathbf{y},t)\,\ddot{\mathbf{y}} + \mathbf{g}(\dot{\mathbf{y}},\mathbf{y},t) = 0 \qquad (2.17)$$

where $\mathbf{M}$ is the $f \times f$ -symmetric mass matrix and $\mathbf{g}$ is a $f \times 1$ -vector function.

Lagrange's equation

Lagrange's equation reads for a system of p rigid bodies as

$$\frac{d}{dt}\,\frac{\partial T}{\partial \dot{\mathbf{y}}} + \frac{\partial T}{\partial \mathbf{y}} = \mathbf{q}\ , \qquad (2.18)$$

where

$$T = \frac{1}{2}\sum_{i=1}^{p}\left(\mathbf{v}_i^T\,m_i\,\mathbf{v}_i + \boldsymbol{\omega}_i^T\,\mathbf{I}_i\,\boldsymbol{\omega}_i\right) \qquad (2.19)$$

is the scalar kinetic energy and

$$q = \sum_{i=1}^{p} \left(\mathfrak{J}_{Ti}^{T} \mathbf{f}_i + \mathfrak{J}_{Ri}^{T} \mathbf{l}_i \right) \tag{2.20}$$

is the generalized $f \times 1$ -force vector.

As simple as Lagrange's equation is looking as difficult may be the evaluation. This is obvious if (2.5) is introduced in (2.19)

$$\begin{aligned} T = \frac{1}{2} \sum_{i=1}^{p} \Big(& \dot{\mathbf{y}}^{T} \mathfrak{J}_{Ti}^{T} m_i \mathfrak{J}_{Ti} \dot{\mathbf{y}} + 2 \dot{\mathbf{y}} \mathfrak{J}_{Ti} m_i \bar{\mathbf{v}}_i + \bar{\mathbf{v}}_i^{T} m_i \bar{\mathbf{v}}_i + \\ & + \dot{\mathbf{y}}^{T} \mathfrak{J}_{Ri}^{T} \mathbf{I}_i \mathfrak{J}_{Ri} \dot{\mathbf{y}} + 2 \dot{\mathbf{y}} \mathfrak{J}_{Ri} \mathbf{I}_i \bar{\boldsymbol{\omega}}_i + \bar{\boldsymbol{\omega}}_i^{T} \mathbf{I}_i \bar{\boldsymbol{\omega}}_i \Big) . \end{aligned} \tag{2.21}$$

However, after proceeding through the differentiation of the kinetic energy (2.21), it finally follows from Lagrange's equation (2.18) exactly the same equation of motion (2.17) as obtained via Newton's and Euler's equation. In recent days of digital and electronic computers, therefore, Newton's and Euler's equation seem to be more convenient since only matrix operations are required. Further, Euler's and Newton's equation can be easily extended to moving reference frames (relative motion) as shown by Schiehlen (1972) .

Linearization

Assume equilibrium position $\mathbf{y} = \mathbf{0}$ and small oscillations in the neighborhood of the equilibrium position. Then, the second and higher order terms in the generalized coordinates can be