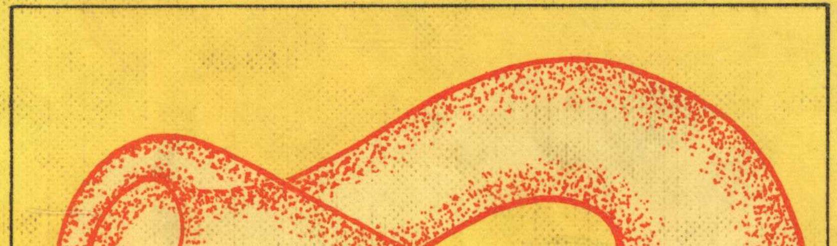
Undergraduate Texts in Mathematics

Fred H. Croom

Basic Concepts of Algebraic Topology



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Preface

This text is intended as a one semester introduction to algebraic topology at the undergraduate and beginning graduate levels. Basically, it covers simplicial homology theory, the fundamental group, covering spaces, the higher homotopy groups and introductory singular homology theory.

The text follows a broad historical outline and uses the proofs of the discoverers of the important theorems when this is consistent with the elementary level of the course. This method of presentation is intended to reduce the abstract nature of algebraic topology to a level that is palatable for the beginning student and to provide motivation and cohesion that are often lacking in abstact treatments. The text emphasizes the geometric approach to algebraic topology and attempts to show the importance of topological concepts by applying them to problems of geometry and analysis.

The prerequisites for this course are calculus at the sophomore level, a one semester introduction to the theory of groups, a one semester introduction to point-set topology and some familiarity with vector spaces. Outlines of the prerequisite material can be found in the appendices at the end of the text. It is suggested that the reader not spend time initially working on the appendices, but rather that he read from the beginning of the text, referring to the appendices as his memory needs refreshing. The text is designed for use by college juniors of normal intelligence and does not require "mathematical maturity" beyond the junior level.

The core of the course is the first four chapters—geometric complexes, simplicial homology groups, simplicial mappings, and the fundamental group. After completing Chapter 4, the reader may take the chapters in any order that suits him. Those particularly interested in the homology sequence and singular homology may choose, for example, to skip Chapter 5 (covering spaces) and Chapter 6 (the higher homotopy groups) temporarily and proceed directly to Chapter 7. There is not so much material here, however, that the instructor will have to pick and choose in order to cover something in every chapter. A normal class should complete the first six chapters and get well into Chapter 7.

No one semester course can cover all areas of algebraic topology, and many important areas have been omitted from this text or passed over with only brief mention. There is a fairly extensive list of references that will point the student to more advanced aspects of the subject. There are, in addition, references of historical importance for those interested in tracing concepts to their origins. Conventional square brackets are used in referring to the numbered items in the bibliography.

For internal reference, theorems and examples are numbered consecutively within each chapter. For example, "Theorem IV.7" refers to Theorem 7 of Chapter 4. In addition, important theorems are indicated by their names in the mathematical literature, usually a descriptive name (e.g., Theorem 5.4, The Covering Homotopy Property) or the name of the discoverer (e.g., Theorem 7.8, The Lefschetz Fixed Point Theorem.)

A few advanced theorems, the Freudenthal Suspension Theorem, the Hopf Classification Theorem, and the Hurewicz Isomorphism Theorem, for example, are stated in the text without proof. Although the proofs of these results are too advanced for this course, the statements themselves and some of their applications are not. Students at the beginning level of algebraic topology can appreciate the beauty and power of these theorems, and seeing them without proof may stimulate the reader to pursue them at a more advanced level in the literature. References to reasonably accessible proofs are given in each case.

The notation used in this text is fairly standard, and a real attempt has been made to keep it as simple as possible. A list of commonly used symbols with definitions and page references follows the table of contents. The end of each proof is indicated by a hollow square, $\square$.

There are many exercises of varying degrees of difficulty. Only the most extraordinary student could solve them all on first reading. Most of the problems give standard practice in using the text material or complete arguments outlined in the text. A few provide real extensions of the ideas covered in the text and represent worthy projects for undergraduate research and independent study beyond the scope of a normal course.

I make no claim of originality for the concepts, theorems, or proofs presented in this text. I am indebted to Wayne Patty for introducing me to algebraic topology and to the many authors and research mathematicians whose work I have read and used.

I am deeply grateful to Stephen Puckette and Paul Halmos for their help and encouragement during the preparation of this text. I am also indebted to Mrs. Barbara Hart for her patience and careful work in typing the manuscript.

FRED H. CROOM

Algebraic Topology: An Introduction

by **W. S. Massey**
(Graduate Texts in Mathematics, Vol. 56)
1977. xxi, 261p. 61 illus. cloth

Here is a lucid examination of algebraic topology, designed to introduce advanced undergraduate or beginning graduate students to the subject as painlessly as possible. *Algebraic Topology: An Introduction* is the first textbook to offer a straight-forward treatment of "standard" topics such as 2-dimensional manifolds, the fundamental group, and covering spaces. The author's exposition of these topics is stripped of unnecessary definitions and terminology and complemented by a wealth of examples and exercises.

Algebraic Topology: An Introduction evolved from lectures given at Yale University to graduate and undergraduate students over a period of several years. The author has incorporated the questions, criticisms and suggestions of his students in developing the text. The prerequisites for its study are minimal: some group theory, such as that normally contained in an undergraduate algebra course on the junior-senior level, and a one-semester undergraduate course in general topology.

Lectures on Algebraic Topology

by **A. Dold**
(Grundlehren der mathematischen Wissenschaften, Vol. 200)
1972. xi, 377p. 10 illus. cloth

Lectures on Algebraic Topology presents a comprehensive examination of singular homology and cohomology, with special emphasis on products and manifolds. The book also contains chapters on chain complexes and homological algebra, applications of homology to the geometry of euclidean space, and CW-spaces.

Developed from a one-year course on algebraic topology, *Lectures on Algebraic Topology* will serve admirably as a text for the same. Its appendix contains the presentation of Kan- and Čech-extensions of functors as a vital tool in algebraic topology. In addition, the book features a set of exercises designed to provide practice in the concepts advanced in the main text, as well as to point out further results and developments.

From the reviews:

"This is a thoroughly modern book on algebraic topology, well suited to serve as a text for university courses, and highly to be recommended to any serious student of modern algebraic topology."

Publicationes Mathematicae

Other Undergraduate Texts in Mathematics

Apostol: Introduction to Analytic Number Theory.
1976. xii, 338 pages.

Chung: Elementary Probability Theory with Stochastic Processes.
1975. x, 325 pages. 36 illus.

Fleming: Functions of Several Variables. Second edition.
1977. xi, 411 pages. 96 illus.

Halmos: Finite-Dimensional Vector Spaces. Second edition.
1974. viii, 200 pages.

Halmos: Naive Set Theory.
1974. vii, 104 pages.

Kemeny/Snell: Finite Markov Chains.
1976. ix, 210 pages.

LeCuyer: Introduction to College Mathematics
Using A Programming Language.
1978. Approximately 450 pages. 150 illus.

Protter/Morrey: A First Course in Real Analysis.
1977. xii, 507 pages. 135 illus.

Sigler: Algebra.
1976. xii, 419 pages. 32 illus.

Singer/Thorpe: Lecture Notes on Elementary Topology and Geometry.
1976. viii, 232 pages. 109 illus.

Smith: Linear Algebra.
1978. ix, 278 pages. 14 illus.

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List of Symbols

$\in$	element of 155
$\notin$	not an element of 155
$\subset$	contained in or subset of 155
$=$	equals
$\neq$	not equal to
$\emptyset$	empty set 155
$\{x : \ldots\}$	set of all x such that ... 155
$\cup$	union of sets 155
$\cap$	intersection of sets 155
$\bar{A}$	closure of a set 158
$X \backslash A$	complement of a set 155
$A \times B$, ΠX_α	product of sets 155, 157
$\|x\|$	absolute value of a real or complex number
$\\|x\\|$	Euclidean norm 161
$\mathbb{R}$	the real line 162
$\mathbb{R}^n$	n-dimensional Euclidean space 161
$\mathbb{C}$	the complex plane 69
B^n	n-dimensional ball 162
S^n	n-dimensional sphere 162
$(x_1, x_2, \ldots, x_n)$	n-tuple 155
$f: X \to Y$	function from X to Y 156
$gf: X \to Z$	composition of functions 156
$f\|_C$	restriction of a function 157
$f(A)$	image of a set 156
$f^{-1}(B)$	inverse image of a set 160
$f^{-1}(y)$	inverse image of a point
f^{-1}	inverse function 156
$<, \leq$	less than, less than or equal to

List of Symbols

$>, \geq$	greater than, greater than or equal to
$\{0\}$	trivial group consisting only of an identity element 163
(a,b)	open interval
$[a,b]$	closed interval
I	closed unit interval [0, 1] 162
I^n	n-dimensional unit cube 106, 162
∂I^n	point set boundary of I^n 106, 162
X/A	quotient space 161
$\cong$	isomorphism 164
σ^n, τ^n	n-simplexes 8
$\dot{\sigma}$	barycenter of a simplex 46
$\lvert K\rvert$	polyhedron associated with a complex K; the geometric carrier of K 10
$K^{(1)}$	first barycentric subdivision of a complex K 47
$K^{(n)}$	nth barycentric subdivision of a complex K 47
$\langle v_0 \ldots v_n \rangle$	n-simplex with vertices $v_0, \ldots, v_n$ 9
$\mathrm{st}(v)$	star of a vertex 43
$\mathrm{ost}(v)$	open star of a vertex 43
$\mathrm{Cl}(\sigma)$	closure of a simplex 10
$[\sigma^p, \sigma^{p-1}]$	incidence number 13
$\alpha \sim x_0 \beta$	loops equivalent modulo x_0 61, 106
$B_p(\ )$	p-dimensional boundary group 18
$C_p(\ )$	p-dimensional chain group 16
$H_p(\ )$	p-dimensional homology group 19
$R_p(\ )$	pth Betti number 26
$Z_p(\ )$	p-dimensional cycle group 18
$\lambda(f)$	Lefschetz number of a map 136, 138
$\pi_1(\ )$	fundamental group 63
$\pi_n(\ )$	nth homotopy group 107
χ	Euler characteristic 27
$\partial(c_p)$	boundary of a p-chain 17
$\partial : C_p(K) \rightarrow C_{p-1}(K)$	boundary homomorphism on chain groups 17
$\partial : H_p(K/L) \rightarrow H_{p-1}(L)$	boundary homomorphism on homology groups 142
$g \cdot \sigma^p$	an elementary p-chain 16
Σ	sum
diam	diameter 159
dim	dimension 26
$A = (a_{ij})$	matrix 167
exp, e^z	the exponential function on the complex plane 69, 85
sin	the sine function
cos	the cosine function
$\oplus$	direct sum of groups 165
$\mathbb{Z}$	the additive group of integers 164

Geometric Complexes and Polyhedra 1

1.1 Introduction

Topology is an abstraction of geometry; it deals with sets having a structure which permits the definition of continuity for functions and a concept of "closeness" of points and sets. This structure, called the "topology" on the set, was originally determined from the properties of open sets in Euclidean spaces, particularly the Euclidean plane.

It is assumed in this text that the reader has some familiarity with basic topology, including such concepts as open and closed sets, compactness, connectedness, metrizability, continuity, and homeomorphism. All of these are normally studied in what is called "point-set topology"; an outline of the prerequisite information is contained in Appendix 2.

Point-set topology was strongly influenced by the general theory of sets developed by Georg Cantor around 1880, and it received its primary impetus from the introduction of general metric spaces by Maurice Frechet in 1906 and the appearance of the book *Grundzüge der Mengenlehre* by Felix Hausdorff in 1912.

Although the historical origins of algebraic topology were somewhat different, algebraic topology and point-set topology share a common goal: to determine the nature of topological spaces by means of properties which are invariant under homeomorphisms. Algebraic topology describes the structure of a topological space by associating with it an algebraic system, usually a group or a sequence of groups. For a space X, the associated group $G(X)$ reflects the geometric structure of X, particularly the arrangement of the "holes" in the space. There is a natural interplay between continuous maps $f: X \to Y$ from one space to another and algebraic homomorphisms $f^*: G(X) \to G(Y)$ on their associated groups.

Consider, for example, the unit circle S^1 in the Euclidean plane. The circle has one hole, and this is reflected in the fact that its associated group is generated by one element. The space composed of two tangent circles (a figure eight) has two holes, and its associated group requires two generating elements.

The group associated with any space is a topological invariant of that space; in other words, homeomorphic spaces have isomorphic groups. The groups thus give a method of comparing spaces. In our example, the circle and figure eight are not homeomorphic since their associated groups are not isomorphic.

Ideally, one would like to say that any topological spaces sharing a specified list of topological properties must be homeomorphic. Theorems of this type are called *classification theorems* because they divide topological spaces into classes of topologically equivalent members. This is the sort of theorem to which topology aspires, thus far with limited success. The reader should be warned that an isomorphism between groups does not, in general, guarantee that the associated spaces are homeomorphic.

There are several methods by which groups can be associated with topological spaces, and we shall examine two of them, *homology* and *homotopy*, in this course. The purpose is the same in each case: to let the algebraic structure of the group reflect the topological and geometric structures of the underlying space. Once the groups have been defined and their basic properties established, many beautiful geometric theorems can be proved by algebraic arguments. The power of algebraic topology is derived from its use of algebraic machinery to solve problems in topology and geometry.

The systematic study of algebraic topology was initiated by the French mathematician Henri Poincaré (1854–1912) in a series of papers[1] during the years 1895–1901. Algebraic topology, or analysis situs, did not develop as a branch of point-set topology. Poincaré's original paper predated Frechet's introduction of general metric spaces by eleven years and Hausdorff's classic treatise on point-set topology, *Grundzüge der Mengenlehre*, by seventeen years. Moreover, the motivations behind the two subjects were different. Point-set topology developed as a general, abstract theory to deal with continuous functions in a wide variety of settings. Algebraic topology was motivated by specific geometric problems involving paths, surfaces, and geometry in Euclidean spaces. Unlike point-set topology, algebraic topology was not an outgrowth of Cantor's general theory of sets. Indeed, in an address to the International Mathematical Congress of 1908, Poincaré referred to point-set theory as a "disease" from which future generations would recover.

Poincaré shared with David Hilbert (1862–1943) the distinction of being the leading mathematician of his time. As we shall see, Poincaré's geometric

[1] The papers were *Analysis Situs, Complément à l'Analysis Situs, Deuxième Complément*, and *Cinquième Complément*. The other papers in this sequence, the third and fourth complements, deal with algebraic geometry.

insight was nothing short of phenomenal. He made significant contributions in differential equations (his original specialty), complex variables, algebra, algebraic geometry, celestial mechanics, mathematical physics, astronomy, and topology. He wrote thirty books and over five hundred papers on new mathematics. The volume of Poincaré's mathematical works is surpassed only by that of Leonard Euler's. In addition, Poincaré was a leading writer on popular science and philosophy of mathematics.

In the remaining sections of this chapter we shall examine some of the types of problems that led to the introduction of algebraic topology and define polyhedra, the class of spaces to which homology groups will be applied in Chapter 2.

1.2 Examples

The following are offered as examples of the types of problems that led to the development of algebraic topology by Poincaré. They are hard problems, but the reader who has not studied them before has no cause for alarm. We will use them only to illustrate the mathematical climate of the 1890's and to motivate Poincaré's fundamental ideas.

1.2.1 *The Jordan Curve Theorem and Related Problems*

The French mathematician Camille Jordan (1858–1922) was first to point out that the following "intuitively obvious" fact required proof, and the resulting theorem has been named for him.

Jordan Curve Theorem. *A simple closed curve C (i.e., a homeomorphic image of a circle) in the Euclidean plane separates the plane into two open connected sets with C as their common boundary. Exactly one of these open connected sets (the "inner region") is bounded.*

Jordan proposed this problem in 1892, but it was not solved by him. That distinction belongs to Oswald Veblen (1880–1960), one of the guiding forces in the development of algebraic topology, who published the first correct solution in 1905 [55].

Lest the reader be misguided by his intuition, we present the following related conjecture which was also of interest at the turn of the century.

Conjecture. *Suppose D is a subset of the Euclidean plane* $\mathbb{R}^2$ *and is the boundary of each component of its complement* $\mathbb{R}^2 \setminus D$*. If* $\mathbb{R}^2 \setminus D$ *has a bounded component, then D is a simple closed curve.*

This conjecture was proved false by L. E. J. Brouwer (1881–1966) at about the same time that Veblen gave the first correct proof of the Jordan Curve Theorem. The following counterexample is due to the Japanese geometer Yoneyama (1917) and is known as the Lakes of Wada.

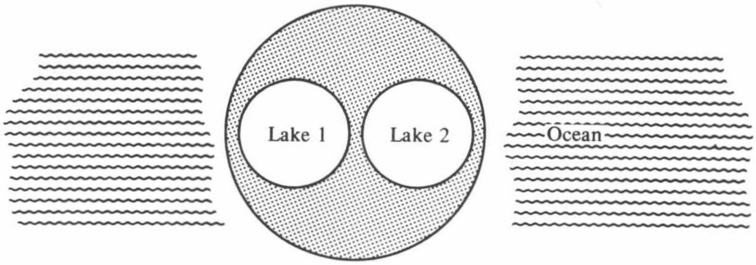

Figure 1.1

Consider the double annulus in Figure 1.1 as an island with two lakes having water of distinct colors surrounded by the ocean. By constructing canals from the ocean and the lakes into the island, we shall define three connected open sets. First, canals are constructed bringing water from the sea and from each lake to within distance $d = 1$ of each dry point of the island. This process is repeated for $d = \frac{1}{2}, \frac{1}{4}, \ldots, (\frac{1}{2})^n, \ldots$, with no intersection of canals. The two lakes with their canal systems and the ocean with its canal form three regions in the plane with the remaining "dry land" D as common boundary. Since D separates the plane into three connected open sets instead of two, the Jordan Curve Theorem shows that D is not a simple closed curve.

1.2.2 *Integration on Surfaces and Multiply-connected Domains*

Consider the annulus in Figure 1.2 enclosed between the two circles H and K.

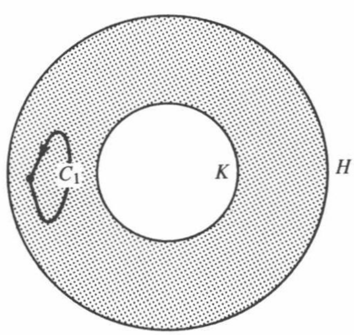

Figure 1.2

We are interested in evaluating curve integrals

$$\int_C p\,dx + q\,dy$$

where $p = p(x, y)$ and $q = q(x, y)$ are continuous functions of two variables whose partial derivatives are continuous and satisfy the relation

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}.$$

Since curve C_1 can be continuously deformed to a point in the annulus, then

$$\int_{C_1} p\,dx + q\,dy = 0.$$

Thus C_1 is considered to be negligible as far as curve integrals are concerned, and we say that C_1 is "equivalent" to a constant path.

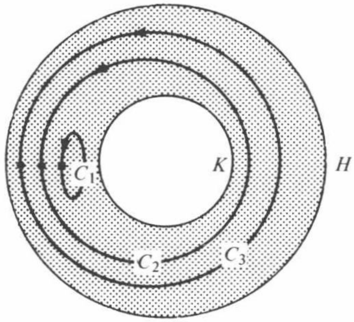

Figure 1.3

Green's Theorem insures that the integrals over curves C_2 and C_3 of Figure 1.3 are equal, so we can consider C_2 and C_3 to be "equivalent."

How can we give a more precise meaning to this idea of equivalence of paths? There are several possible ways, and two of them form the basic ideas of algebraic topology. First, we might consider C_2 and C_3 equivalent because each can be transformed continuously into the other within the annulus. This is the basic idea of homotopy theory, and we would say that C_2 and C_3 are *homotopic paths*. Curve C_1 is homotopic to a trivial (or constant) path since it can be shrunk to a point. Note that C_2 and C_1 are not homotopic paths since C_2 cannot be pulled across the "hole" that it encloses. For the same reason, C_1 is not homotopic to C_3.

Another approach is to say that C_2 and C_3 are equivalent because they form the boundary of a region enclosed in the annulus. This second idea is the basis of homology theory, and C_2 and C_3 would be called *homologous paths*. Curve C_1 is *homologous to zero* since it is the entire boundary of a region enclosed in the annulus. Note that C_1 is not homologous to either C_2 or C_3.

The ideas of homology and homotopy were introduced by Poincaré in his original paper *Analysis Situs* [49] in 1895. We shall consider both topics in some detail as the course progresses.

1.2.3 *Classification of Surfaces and Polyhedra*

Consider the problem of explaining the difference between a sphere S^2 and a torus T as shown in Figure 1.4. The difference, of course, is apparent: the sphere has one hole, and the torus has two. Moreover, the hole in the sphere is somehow different from those in the torus. The problem is to explain this difference in a mathematically rigorous way which can be applied to more complicated and less intuitive examples.