MEMOIRS
of the
American Mathematical Society

Volume 231 • Number 1084 (first of 5 numbers) • September 2014

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Providence, Rhode Island

Library of Congress Cataloging-in-Publication Data

Weiss, Michael S., 1955-

Automorphisms of manifolds and algebraic K-theory: Part III / Michael S. Weiss, Bruce E. Williams.

volumes cm. – (Memoirs of the American Mathematical Society, issn 0065-9266 ; volume 231, number 1084)

Description based on: pt. 3.

Includes bibliographical references.

Contents: – pt. 3. [Without special title]

ISBN 978-1-4704-0981-4 (alk. paper : pt. 3)

1. Homology theory. 2. Manifolds (Mathematics) 3. K-theory. 4. Automorphisms. I. Williams, Bruce E., 1945- II. Title.

QA612.7.W38 2014

514′.34–dc23 2014015617

DOI: http://dx.doi.org/10.1090/memo/1084

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Subscription information. Beginning with the January 2010 issue, *Memoirs* is accessible from www.ams.org/journals. The 2014 subscription begins with volume 227 and consists of six mailings, each containing one or more numbers. Subscription prices are as follows: for paper delivery, US$827 list, US$661.60 institutional member; for electronic delivery, US$728 list, US$582.40 institutional member. Upon request, subscribers to paper delivery of this journal are also entitled to receive electronic delivery. If ordering the paper version, add US$10 for delivery within the United States; US$69 for outside the United States. Subscription renewals are subject to late fees. See www.ams.org/help-faq for more journal subscription information. Each number may be ordered separately; *please specify number* when ordering an individual number.

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Subscriptions and orders should be addressed to the American Mathematical Society, P. O. Box 845904, Boston, MA 02284-5904 USA. *All orders must be accompanied by payment.* Other correspondence should be addressed to 201 Charles Street, Providence, RI 02904-2294 USA.

Memoirs of the American Mathematical Society (ISSN 0065-9266 (print); 1947-6221 (online)) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA.

This publication is indexed in *Mathematical Reviews*®, *Zentralblatt MATH*, *Science Citation Index*®, *Science Citation Index*™*-Expanded*, *ISI Alerting Services*℠, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®*/Physical, Chemical & Earth Sciences*. This publication is archived in *Portico* and *CLOCKSS*.
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10 9 8 7 6 5 4 3 2 1 19 18 17 16 15 14

Abstract

The structure space $\mathcal{S}(M)$ of a closed topological m-manifold M classifies bundles whose fibers are closed m-manifolds equipped with a homotopy equivalence to M. We construct a highly connected map from $\mathcal{S}(M)$ to a concoction of algebraic L-theory and algebraic K-theory spaces associated with M. The construction refines the well-known surgery theoretic analysis of the block structure space of M in terms of L-theory.

Received by the editor September 28, 2012, and, in revised form, October 9, 2012.

Article electronically published on January 14, 2014.

DOI: `http://dx.doi.org/10.1090/memo/1084`

2010 *Mathematics Subject Classification.* Primary 57R65, 18F25; secondary 55R10, 57R22.

Affiliations at time of publication: Michael S. Weiss, Mathematisches Institut, Universität Münster, Einsteinstrasse 62, 48149 Münster, Germany, email: m.weiss@uni-muenster.de; and Bruce E. Williams, Department of Mathematics, University of Notre Dame, Notre Dame, Indiana 46556-5683, USA, email: williams.4@nd.edu.

Contents

CHAPTER 1

Introduction

The structure space $\mathcal{S}(M)$ of a closed topological m-manifold M is the classifying space for bundles $E \to X$ with an arbitrary CW-space X as base, closed topological manifolds as fibers and with a fiber homotopy trivialization

$$E \to M \times X$$

(a homotopy equivalence and a map over X). The points of $\mathcal{S}(M)$ can loosely be imagined as pairs (N, f) where N is a closed m-manifold and $f\colon N \to M$ is a homotopy equivalence. To explain the relationship between $\mathcal{S}(M)$ and automorphisms of M, we invoke $\mathcal{H}om(M)$, the topological group of homeomorphisms from M to M, and $\mathrm{G}(M)$, the grouplike topological monoid of homotopy equivalences from M to M. In practice we work with simplicial models of $\mathcal{H}om(M)$ and $\mathrm{G}(M)$. The homotopy fiber of the inclusion $B\mathcal{H}om(M) \to B\mathrm{G}(M)$ is homotopy equivalent to a union of connected components of $\mathcal{S}(M)$.

The main result of this paper is a calculation of the homotopy type of $\mathcal{S}(M)$ in the so-called concordance stable range, in terms of L- and algebraic K-theory. With m fixed as above, we construct a homotopy invariant functor

$$(Y, \xi) \mapsto \mathbf{LA}_{\bullet\%}(Y, \xi, m)$$

from spaces Y with spherical fibrations ξ to spectra. The spectrum $\mathbf{LA}_{\bullet\%}(Y, \xi, m)$ is a concoction of the L-theory and the algebraic K-theory of spaces [**27**] associated with Y, compounded with an assembly construction [**21**]. (The subscript $\%$ is for homotopy fibers of assembly maps.) In the case where $Y = M$ (nonempty and connected for simplicity) and ξ is ν, the normal fibration of M, there is a "local degree" map

$$\Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(M, \nu, m) \longrightarrow 8\mathbb{Z} \subset \mathbb{Z}.$$

There is then a highly connected map

$$\mathcal{S}(M) \longrightarrow \mathrm{fiber}\big[\,\Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(M, \nu, m) \xrightarrow{\text{local deg.}} 8\mathbb{Z}\,\big] \tag{1.1}$$

where fiber in this case means the fiber over $0 \in 8\mathbb{Z}$, an infinite loop space. The connectivity estimate is given by the concordance stable range. In practice that translates into $m/3$ approximately, but in theory it is more convoluted and the reader is referred to definition 11.5. The result has a generalization to the case in which M is compact with nonempty boundary. It looks formally the same. Points of $\mathcal{S}(M)$ can be imagined as pairs (N, f) where N is a compact manifold with boundary and $f\colon (N, \partial N) \to (M, \partial M)$ is a homotopy equivalence of pairs restricting to a homeomorphism of ∂N with ∂M.

We now give a slightly more detailed, although still sketchy, definition of the spectrum $\Omega^m\mathbf{LA}_{\bullet\%}(Y, \xi, m)$. (Details can be found in chapter 9.) It is the total homotopy fiber of a commutative square

$$
\begin{array}{ccc}
\Omega^m \mathbf{L}_\bullet{}^{\%}(Y,\xi) & \longrightarrow & S^1 \wedge \mathbf{A}^{\%}(Y,\xi,m)_{h\mathbb{Z}/2} \\
\downarrow & & \downarrow \\
\Omega^m \mathbf{L}_\bullet(Y,\xi) & \longrightarrow & S^1 \wedge \mathbf{A}(Y,\xi,m)_{h\mathbb{Z}/2}\ .
\end{array}
\tag{1.2}
$$

The left-hand column is the quadratic L-theory assembly map. (This has a variety of equivalent descriptions and in the simplest of these it depends only on Y and the orientation double covering w_ξ of Y determined by ξ. We do not insist on such a simple description because that would make the horizontal maps in the diagram more obscure; therefore $\mathbf{L}_\bullet(Y,\xi)$ rather than $\mathbf{L}_\bullet(Y,w_\xi)$ is the notation which we prefer.) The right-hand column is the Waldhausen A-theory assembly map with $S^1\wedge$ and homotopy orbit construction inflicted. Both columns use a category of (finitely dominated) retractive spaces or spectra over Y, subject to finiteness conditions and equipped with a notion of Spanier-Whitehead duality which depends on ξ and m. The horizontal maps are variants of a natural transformation Ξ which was defined in [**37**]; the precise relationship will be clarified in a moment. By opting for finitely dominated retractive spaces in both columns we have implicitly selected the decoration p. (The infinite loop space $\Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(Y,\xi,m)$ is decoration independent, i.e., any consistent choice of decoration from the list $h,p,\ldots,\langle -i\rangle,\ldots,\langle -\infty\rangle$ gives the same result up to a homotopy equivalence.)

This is a definition which relates $\Omega^m\mathbf{LA}_{\bullet\%}(Y,\xi,m)$ to known and trusted concepts in algebraic L- and K-theory. For our constructions we prefer another definition of $\Omega^m\mathbf{LA}_{\bullet\%}(Y,\xi,m)$ as the homotopy fiber of the map between homotopy pullbacks of the rows in the commutative diagram

$$
\begin{array}{ccccc}
\Omega^m \mathbf{VL}^{\bullet\%}(Y,\xi) & \xrightarrow{\ \Xi\ } & \mathbf{A}^{\%}(Y,\xi,m)^{th\mathbb{Z}/2} & \xleftarrow{\text{incl.}} & \mathbf{A}^{\%}(Y,\xi,m)^{h\mathbb{Z}/2} \\
\downarrow & & \downarrow & & \downarrow \\
\Omega^m \mathbf{VL}^{\bullet}(Y,\xi) & \xrightarrow{\ \Xi\ } & \mathbf{A}(Y,\xi,m)^{th\mathbb{Z}/2} & \xleftarrow{\text{incl.}} & \mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2}
\end{array}
\tag{1.3}
$$

where the left-hand column is the assembly map in a form of *visible symmetric* L-theory. The visible symmetric L-theory will be reviewed in chapter 3. There are forgetful natural transformations

$$\mathbf{L}_\bullet(Y,\xi) \longrightarrow \mathbf{VL}^\bullet(Y,\xi)$$

which fit into a homotopy cartesian square

$$
\begin{array}{ccc}
\mathbf{L}_\bullet^{\%}(Y,\xi) & \longrightarrow & \mathbf{VL}^{\bullet\%}(Y,\xi) \\
\downarrow & & \downarrow \\
\mathbf{L}_\bullet(Y,\xi) & \longrightarrow & \mathbf{VL}^{\bullet}(Y,\xi).
\end{array}
$$

This will also be reviewed in chapter 3. Together with the norm fibration sequence

$$S^1\wedge \mathbf{A}(Y,\xi,m)_{h\mathbb{Z}/2} \longleftarrow \mathbf{A}(Y,\xi,m)^{th\mathbb{Z}/2} \xleftarrow{\text{incl.}} \mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2}$$

(and a variant with $\mathbf{A}^{\%}$ instead of $\mathbf{A}$), this explains why the two competing definitions of $\mathbf{LA}_{\bullet\%}(Y,\xi,m)$, relying on diagrams (1.2) and (1.3) respectively, are consistent.

Our reasons for preferring the second definition of $\mathbf{LA}_{\bullet\%}$ are strategic. Quadratic L-theory famously serves as a receptacle for relative invariants, such as surgery obstructions of degree one normal *maps* $X \to Y$. By contrast, visible symmetric L-theory is a mild refinement of symmetric L-theory and as such a good receptacle for absolute invariants: generalized signatures of Poincaré duality spaces, say. In particular a Poincaré duality space Y of formal dimension m, and with Spivak normal fibration ξ, determines a characteristic element

$$v_L(Y) \in \Omega^{\infty+m}\mathbf{VL}^\bullet(Y,\xi)$$

which can be viewed as a refined signature. (We think of it as a point in an infinite loop space, not a connected component of an infinite loop space.) Refining this some more to pick up algebraic K-theory information, we get

$$\sigma(Y) \in \Omega^{\infty+m}\mathbf{VLA}^\bullet(Y,\xi,m) \tag{1.4}$$

where

$$\Omega^m\mathbf{VLA}^\bullet(Y,\xi,m) := \operatorname{holim}\left(\begin{array}{ccc} & & \mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2} \\ & & \big\downarrow{\scriptstyle \text{inclusion}} \\ \Omega^m\mathbf{VL}^\bullet(Y,\xi) & \xrightarrow{\ \Xi\ } & \mathbf{A}(Y,\xi,m)^{th\mathbb{Z}/2} \end{array}\right).$$

This refinement expresses a compatibility between $v_L(Y)$ and the self-dual Euler characteristic

$$v_K(Y) \in \Omega^\infty(\mathbf{A}(Y,\xi,m)^{h\mathbb{Z}/2}) .$$

We construct $\sigma(Y)$ in chapter 9. The construction enjoys continuity properties. It can be applied to the fibers of a fibration $E \to B$ whose fibers are Poincaré duality spaces E_b of formal dimension m, so that we obtain a section of a fibration on B whose fibers are certain infinite loop spaces.

Now suppose that the Poincaré duality space Y is a closed manifold of dimension m. Then the point $\sigma(Y)$ lifts across the visible L-theory and A-theory assembly maps to a point

$$\sigma^\%(Y) \in \Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(Y,\xi,m) . \tag{1.5}$$

We construct $\sigma^\%(Y)$ in chapter 10. Again this construction enjoys continuity properties: it can be applied it to the fibers of a fibre bundle $E \to B$ whose fibers are closed manifolds E_b of dimension m, so that we obtain a section of a fibration on B whose fibers are certain infinite loop spaces. (This is very hard to establish, like the continuity property of excisive Euler characteristics in [**10**]. Relying on [**16**], we reduce to the case of fiber bundles with discrete structure group.)

In particular, the space $\mathcal{S}(M)$ carries a universal bundle $E \to \mathcal{S}(M)$ of closed manifolds with a fiber homotopy trivialization $E \simeq \mathcal{S}(M) \times M$. Therefore each point $(N,f) \in \mathcal{S}(M)$ determines an element $f_*\sigma^\%(N) \in \Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(M,\nu_M,m)$, whose image in $\Omega^{\infty+m}\mathbf{VLA}^\bullet(M,\nu_M,m)$ under assembly comes with a preferred path to $\sigma(M) \in \Omega^{\infty+m}\mathbf{VLA}^\bullet(M,\nu_M,m)$. This gives us the map (1.1): here it comes as a map from $\mathcal{S}(M)$ to the homotopy fiber, over the point $\sigma(M)$, of the assembly map

$$\Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(M,\nu_M,m) \longrightarrow \Omega^{\infty+m}\mathbf{VLA}^\bullet(M,\nu_M,m) .$$

Similar ideas, i.e., a firework of characteristics and signatures, can be used to show that the map (1.1) is highly connected; we give an overview in chapter 2 before developing the details.

This result has many precursors. The most fundamental and best known of these belong to surgery theory. From the surgery point of view it is very natural to introduce certain "block" structure spaces such as

$$\widetilde{\mathcal{S}}^s(M)\,,\quad \widetilde{\mathcal{S}}^h(M)\,.$$

These are designed in such a way that $\pi_0\widetilde{\mathcal{S}}^s(M)$ and $\pi_0\widetilde{\mathcal{S}}^h(M)$ are identifiable with, respectively, the subset of $\pi_0\mathcal{S}(M)$ determined by the simple homotopy equivalences, and the quotient set of $\pi_0\mathcal{S}(M)$ determined by the h-cobordism relation. In addition they have the property

$$\pi_i\widetilde{\mathcal{S}}^s(M)\cong\pi_0\widetilde{\mathcal{S}}^s(M\times D^i)\,,\quad \pi_i\widetilde{\mathcal{S}}^h(M)\cong\pi_0\widetilde{\mathcal{S}}^h(M\times D^i)\,.$$

This is obviously very useful in calculations. The surgery-theoretic calculations of these spaces are of the form

$$\begin{aligned}\widetilde{\mathcal{S}}^s(M)\ &\simeq\ \text{fiber}\,[\,\Omega^{\infty+m}\mathbf{L}^s_\bullet{}_\%(M,w)\longrightarrow 8\mathbb{Z}\,]\,,\\ \widetilde{\mathcal{S}}^h(M)\ &\simeq\ \text{fiber}\,[\,\Omega^{\infty+m}\mathbf{L}^h_\bullet{}_\%(M,w)\longrightarrow 8\mathbb{Z}\,]\,,\end{aligned}\tag{1.6}$$

where $\mathbf{L}^s_\bullet{}_\%$ and $\mathbf{L}^h_\bullet{}_\%$ are homotopy invariant functors from spaces with double coverings to spectra. (In particular w denotes the orientation covering of M.) The functors $\mathbf{L}^s_\bullet{}_\%$ and $\mathbf{L}^h_\bullet{}_\%$ can be defined entirely in terms of algebraic L-theory, again compounded with assembly. They are therefore fully 4-periodic:

$$\Omega^4\mathbf{L}^s_\bullet{}_\%(X,v)\simeq\mathbf{L}^s_\bullet{}_\%(X,v)\,,\qquad \Omega^4\mathbf{L}^h_\bullet{}_\%(X,v)\simeq\mathbf{L}^h_\bullet{}_\%(X,v)\,.$$

This calculation of $\widetilde{\mathcal{S}}^s(M)$ and $\widetilde{\mathcal{S}}^h(M)$ is sometimes called the Casson-Sullivan-Wall-Quinn-Ranicki theorem. An earlier version of it, describing the homotopy groups of the block structure space(s), is known as the Casson-Sullivan-Wall long exact sequence. The space level formulation was championed by Quinn. The complete and final reduction to L-theory, at the space level, is mainly due to the untiring efforts of Ranicki. This took many years.

Our calculation of structure spaces $\mathcal{S}(M)$ in the concordance stable range is in agreement with the surgery theoretic calculation of block structure spaces. For example, there is a commutative diagram

$$\begin{array}{ccc}\mathcal{S}^s(M) & \longrightarrow & \text{fiber}[\Omega^{\infty+m}\mathbf{LA}_\bullet{}_\%(M,\nu,m)\longrightarrow \text{Wh}(\pi_1M)\times 8\mathbb{Z}]\\ \big\downarrow{\scriptstyle\text{incl.}} & & \big\downarrow{\scriptstyle\text{forgetful}}\\ \widetilde{\mathcal{S}}^s(M) & \xrightarrow{\ \simeq\ } & \text{fiber}[\Omega^{\infty+m}\mathbf{L}^s_\bullet{}_\%(M,w)\longrightarrow 8\mathbb{Z}].\end{array}$$

where the upper horizontal arrow is the restriction of the map (1.1). Passing to vertical homotopy fibers, we obtain a highly connected map

$$\widetilde{\text{TOP}}(M)/\text{TOP}(M)\longrightarrow\Omega^\infty(\mathbf{A}^s{}_\%(M,\nu,m)_{h\mathbb{Z}/2})\,.$$

This is reminiscent of a highly connected map

$$\widetilde{\mathrm{TOP}}(M)/\mathrm{TOP}(M) \longrightarrow \Omega^\infty(\mathbf{H}^s(M)_{h\mathbb{Z}/2})$$

constructed in [**36**]; see also [**41**] for notation. These two maps are intended to be the same, modulo Waldhausen's identification of the h-cobordism spectrum $\mathbf{H}^s(M)$ with $\mathbf{A}^s{}_\%(M)$. We do not quite prove that here, but we come close to it. It will be the theme of another paper in this series.

Meanwhile Burghelea and Lashof [**6**, cor. D] obtained results on the homotopy type of $\mathcal{S}(M)$. Localizing at odd primes, they were able to construct a highly connected map

$$\Omega\mathcal{S}(M) \;\longrightarrow\; \Omega\widetilde{\mathcal{S}}(M) \times \Omega^{\infty+1}\mathbf{A}_\%(M,\nu,m)^{h\mathbb{Z}/2}\ .$$

(The localization is applied to $\mathcal{S}(M)$, $\widetilde{\mathcal{S}}(M)$ and $\mathbf{A}_\%(M,\nu,m)$ before other operations are carried out: Ω in both sides, $\Omega^{\infty+1}$ and the homotopy fixed point operation in the right-hand side.) After localization of $\mathbf{A}_\%(M,\nu,m)$ at odd primes, the homotopy fixed point spectrum $\mathbf{A}_\%(M,\nu,m)^{h\mathbb{Z}/2}$ is a wedge summand of $\mathbf{A}_\%(M)$ which depends only on ν and the parity of m.

With hindsight, the Burghelea-Lashof result can be explained in terms of our calculation of $\mathcal{S}(M)$ described above and the surgery-theoretic calculation of the block structure space. At odd primes, the six-term diagram (1.3) simplifies because the Tate constructions in the middle column (again to be applied after the localization of $\mathbf{A}_\%(Y,\xi,m)$) vanish. Therefore at odd primes

$$\Omega^m\mathbf{LA}_{\bullet\%}(Y,\xi,m) \;\simeq\; \Omega^m\mathbf{L}_{\bullet\%}(Y,\xi) \vee \mathbf{A}_\%(Y,\xi,m)^{h\mathbb{Z}/2}\ .$$

This paper is a continuation of [**36**] and [**37**]. In another sense it is a continuation of [**10**]. For technical support, we use a fair amount of controlled topology as in [**3**], the Thurston-Mather-McDuff-Segal discrete approximation theory [**16**] for homeomorphism groups as in [**10**], and Spanier-Whitehead duality theory with its implications for algebraic K-theory as in [**39**].

CHAPTER 2

Outline of proof

In the introduction, we gave a rough description of certain invariants of type signature and Euler characteristic for manifolds and Poincaré duality spaces. This led us to a map of the form (1.1). We wish to show that the map is highly connected. The main tools in the proof are

(i) a controlled version of the Casson-Sullivan-Wall-Quinn-Ranicki (CSWQR) theorem in surgery theory;
(ii) more invariants of type signature and Euler characteristic for manifolds and Poincaré duality spaces in a controlled setting;
(iii) a simple downward induction, where the induction beginning relies on (i) while (ii) enables us to do the induction steps.

Let $\mathcal{S}(M\times\mathbb{R}^i\,;\,c)$ be the *controlled* structure space of $M\times\mathbb{R}^i$; here we view $M\times\mathbb{R}^i$ as an open dense subset of the join $M * S^{i-1}$. An element of $\mathcal{S}(M\times\mathbb{R}^i\,;\,c)$ should be thought of as a pair (N, f) where N is a manifold of dimension $m+i$, without boundary, and $f\colon N\to M\times\mathbb{R}^i$ is a *controlled* homotopy equivalence [**3**]. There is also a controlled block structure space

$$\widetilde{\mathcal{S}}^{cs}(M\times\mathbb{R}^i\,;\,c)$$

where the decoration cs (controlled simple) indicates that we allow only structures with vanishing controlled Whitehead torsion.

The homotopy type of $\widetilde{\mathcal{S}}^{cs}(M\times\mathbb{R}^i\,;\,c)$ can be described by a formula which combines the CSWQR ideas with controlled algebra [**3**]: namely,

$$\widetilde{\mathcal{S}}^{cs}(M\times\mathbb{R}^i\,;\,c)\;\simeq\;\text{fiber}\left[\Omega^{\infty+m+i}\mathbf{L}^{cs}_{\bullet}\%(M\times\mathbb{R}^i,\nu\,;\,c)\longrightarrow 8\mathbb{Z}\right]\tag{2.1}$$

where $\mathbf{L}^{cs}_{\bullet}(M\times\mathbb{R}^i,\nu\,;\,c)$ is the controlled quadratic L-theory (with vanishing controlled Whitehead torsion) of the control space $(M*S^{i-1}, M\times\mathbb{R}^i)$. Taking i to the limit we have

$$\operatorname*{colim}_{i\geq 0}\;\widetilde{\mathcal{S}}^{cs}(M\times\mathbb{R}^i\,;\,c)\;\simeq\;\text{fiber}\left[\;\operatorname*{colim}_{i\geq 0}\;\Omega^{\infty+m+i}\mathbf{L}^{cs}_{\bullet}\%(M\times\mathbb{R}^i,\nu\,;\,c)\longrightarrow 8\mathbb{Z}\right]$$

where the colimits are formed using product with $\mathbb{R}$ in various shapes. Moreover, it is well-known [**36**] that the inclusions

$$\operatorname*{colim}_{i\geq 0}\;\mathcal{S}(M\times\mathbb{R}^i\,;\,c)\longleftarrow\;\operatorname*{colim}_{i\geq 0}\;\mathcal{S}^{cs}(M\times\mathbb{R}^i\,;\,c)\longrightarrow\;\operatorname*{colim}_{i\geq 0}\;\widetilde{\mathcal{S}}^{cs}(M\times\mathbb{R}^i\,;\,c)$$

are homotopy equivalences. Therefore we have

$$\operatorname*{colim}_{i\geq 0}\;\mathcal{S}(M\times\mathbb{R}^i\,;\,c)\;\simeq\;\text{fiber}\left[\;\operatorname*{colim}_{i\geq 0}\;\Omega^{\infty+m+i}\mathbf{L}^{cs}_{\bullet}\%(M\times\mathbb{R}^i,\nu\,;\,c)\longrightarrow 8\mathbb{Z}\right]\tag{2.2}$$

and this is the starting point for our downward induction.

Next we discuss the induction steps. Let $(\bar{Y}, Y)$ be a control space. For the present purposes we can take this to mean that $\bar{Y}$ is compact metrizable, and Y is open dense in $\bar{Y}$. A choice of spherical fibration ξ on Y and integer m makes the Waldhausen category of locally finitely dominated retractive spaces over Y into a Waldhausen category with duality (see **[10]** for details). By forming L-theory, K-theory etc., we define spectrum-valued functors

$$(\bar{Y}, Y, \xi) \quad \mapsto \begin{cases} \mathbf{L}_\bullet(Y, \xi\,;\, c), \\ \mathbf{VL}^\bullet(Y, \xi\,;\, c), \\ \mathbf{A}(Y\,;\, c), \\ \mathbf{LA}_\bullet(Y, \xi, m\,;\, c), \\ \mathbf{VLA}^\bullet(Y, \xi, m\,;\, c) \end{cases}$$

much as before. (Three of these can be viewed as functors of a general Waldhausen category with duality; the ones having a **V** in their name use more special features.) The symbol c is a shorthand for control conditions, allowing us to avoid direct reference to the inclusion $Y \to \bar{Y}$. There are natural assembly transformations

$$\begin{aligned} \mathbf{L}_\bullet{}^\%(Y, \xi\,;\, c) &\longrightarrow \mathbf{L}_\bullet(Y, \xi\,;\, c), \\ \mathbf{VL}^{\bullet\%}(Y, \xi\,;\, c) &\longrightarrow \mathbf{VL}^\bullet(Y, \xi\,;\, c), \\ \mathbf{A}^\%(Y\,;\, c) &\longrightarrow \mathbf{A}(Y\,;\, c), \\ \mathbf{LA}_\bullet{}^\%(Y, \xi, m\,;\, c) &\longrightarrow \mathbf{LA}_\bullet(Y, \xi, m\,;\, c), \\ \mathbf{VLA}^{\bullet\%}(Y, \xi, m\,;\, c) &\longrightarrow \mathbf{VLA}^\bullet(Y, \xi, m\,;\, c), \end{aligned} \tag{2.3}$$

where the domain is now designed so that its homotopy groups are the *locally finite* generalized homology groups of Y with (twisted where appropriate) coefficients in $\mathbf{L}_\bullet(\star, \xi)$, $\mathbf{VL}^{\bullet\%}(\star, \xi)$, $\mathbf{A}(\star)$, $\mathbf{LA}_\bullet(\star, \xi, m)$ and $\mathbf{VLA}^{\bullet\%}(\star, \xi, m)$. Here $\star$ should be thought of as a variable point in Y, and we restrict ξ from Y to that point where necessary. The homotopy fibers of the assembly maps (2.3) are denoted by

$$\begin{aligned} & \mathbf{L}_{\bullet\%}(Y, \xi\,;\, c) \\ \simeq\; & \mathbf{VL}^\bullet{}_\%(Y, \xi\,;\, c), \\ & \mathbf{A}_\%(Y\,;\, c), \\ & \mathbf{LA}_{\bullet\%}(Y, \xi, m\,;\, c) \\ \simeq\; & \mathbf{VLA}^\bullet{}_\%(Y, \xi, m\,;\, c), \end{aligned} \tag{2.4}$$

respectively. (The homotopy equivalences asserted here are nontrivial; they are established in chapter 6.) If $(\bar{Y}, Y)$ happens to be a *controlled* Poincaré duality space of formal dimension m and with Spivak normal fibration ξ, then there is a signature invariant

$$\sigma(Y) \in \Omega^{\infty+m}\mathbf{VLA}^\bullet(Y, \xi, m\,;\, c) \tag{2.5}$$

which generalizes (1.4). This invariant has the expected naturality and continuity properties. It is constructed in chapter 9.

If Y happens to be a manifold of dimension m and $\xi = \nu$ is its normal bundle, then $(\bar{Y}, Y)$ is automatically a controlled Poincaré duality space of formal dimension m and the signature invariant $\sigma(Y)$ lifts across the assembly map (2.3) to an element

$$\sigma^\%(Y) \in \Omega^{\infty+m}\mathbf{VLA}^{\bullet\%}(Y, \xi, m), \tag{2.6}$$

generalizing (1.5). This lift is constructed in chapter 10. In particular, the space $\mathcal{S}(M \times \mathbb{R}^i\,;\, c)$ carries a universal bundle where each fiber is an $(m+i)$-manifold N together with a controlled homotopy equivalence $f\colon N \to M \times \mathbb{R}^i$. We may compactify each fiber N to a control space $\bar{N} = N \cup S^{i-1}$ in such a way that N

is open dense in $\bar{N}$ and f extends to a map from $\bar{N}$ to $M * S^{i-1}$. Therefore each point $(N, f) \in \mathcal{S}(M \times \mathbb{R}^i\,;\, c)$ determines an element

$$f_*\sigma^{\%}(N) \in \Omega^{\infty+m+i}\mathbf{VLA}^{\bullet\%}(M \times \mathbb{R}^i, \nu, m+i\,;\, c)$$

whose image in $\Omega^{\infty+m+i}\mathbf{VLA}^{\bullet}(M \times \mathbb{R}^i, \nu, m+i\,;\, c)$ under assembly (2.3) comes with a preferred path to $\sigma(M \times \mathbb{R}^i)$. If this construction were to enjoy certain continuity properties, it would give us a map

$$\mathcal{S}(M \times \mathbb{R}^i\,;\, c) \dashrightarrow \Omega^{\infty+m+i}\mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^i, \nu, m+i\,;\, c)$$

generalizing (1.1), where we think of the target as the homotopy fiber over the point $\sigma(M \times \mathbb{R}^i)$ of the appropriate assembly map in controlled $\mathbf{VLA}^{\bullet}$ theory of $M \times \mathbb{R}^i$. Unfortunately we could not avoid some sacrifices in establishing the continuity properties, and so we only get a map

$$\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i\,;\, c) \longrightarrow \Omega^{\infty+m+i}\mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^i, \nu, m+i\,;\, c) \tag{2.7}$$

where $\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^i\,;\, c) \subset \mathcal{S}(M \times \mathbb{R}^i\,;\, c)$ is the union of the connected components of $\mathcal{S}(M \times \mathbb{R}^i\,;\, c)$ which are reducible in the sense that they come from $\pi_0\mathcal{S}(M)$. Combining the maps (2.7) for all $i \geq 0$ results in a commutative ladder

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i+1}\,;\, c) & \longrightarrow & \Omega^{\infty+m+i+1}\mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i+1}, \nu, m+i+1\,;\, c) \\
\uparrow & & \uparrow \\
\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}^{i}\,;\, c) & \longrightarrow & \Omega^{\infty+m+i}\mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{i}, \nu, m+i\,;\, c) \\
\uparrow & & \uparrow \\
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\mathcal{S}^{\mathrm{rd}}(M \times \mathbb{R}\,;\, c) & \longrightarrow & \Omega^{\infty+m+1}\mathbf{LA}_{\bullet\%}(M \times \mathbb{R}^{1}, \nu, m+1\,;\, c) \\
\uparrow & & \uparrow \\
\mathcal{S}(M) & \longrightarrow & \Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(M, \nu, m)
\end{array} \tag{2.8}$$

where the vertical arrows are given by product with $\mathrm{id}_{\mathbb{R}}$ in the left-hand column, and product with $\sigma^{\%}(\mathbb{R})$ in the right-hand column. Each vertical arrow in the left-hand column induces a surjection on π_0. At the bottom of the ladder we recognize the map (1.1) and at the top we recognize with a small effort (see chapter 13) the map of (2.2). In particular, all homotopy fibers of the horizontal map at the top of the ladder are either contractible or empty. We use downward induction to establish a similar property for *all* horizontal maps in the ladder:

(†) for each of these maps, all homotopy fibers are highly connected or empty.

It is enough show that in each square

$$(2.9)\qquad \begin{array}{ccc} \mathcal{S}^{\mathrm{rd}}(M\times\mathbb{R}^{i+1}\,;\,c) & \longrightarrow & \Omega^{\infty+m+i+1}\mathbf{LA}_{\bullet\%}(M\times\mathbb{R}^{i+1},\nu,m+i+1\,;\,c) \\ \uparrow & & \uparrow \\ \mathcal{S}^{\mathrm{rd}}(M\times\mathbb{R}^{i}\,;\,c) & \longrightarrow & \Omega^{\infty+m+i}\mathbf{LA}_{\bullet\%}(M\times\mathbb{R}^{i},\nu,m+i\,;\,c) \end{array}$$

of the ladder, all total homotopy fibers are highly connected or empty.

Each *vertical* homotopy fiber in the left-hand column can be identified with a union of connected components of a controlled h-cobordism space $\mathcal{H}(N\times\mathbb{R}^i\,;\,c)$, where N is some closed m-manifold homotopy equivalent to M. By an easy calculation carried out mainly in chapter 7, the vertical homotopy fibers in the right-hand column have the form

$$\Omega^\infty\mathbf{A}_\%(M\times\mathbb{R}^i\,;\,c)\ \simeq\ \Omega^\infty\mathbf{A}_\%(N\times\mathbb{R}^i\,;\,c)\ .$$

With these descriptions, the map between matching vertical homotopy fibers in (2.9) extends to a controlled form

$$(2.10)\qquad \mathcal{H}(N\times\mathbb{R}^i\,;\,c)\longrightarrow\Omega^\infty\mathbf{A}_\%(N\times\mathbb{R}^i\,;\,c)$$

of Waldhausen's map relating h-cobordism spaces to A-theory. This is verified in chapter 13. The map (2.10) is highly connected. So all its homotopy fibers are highly connected, and so our claim regarding (2.9) is proved, and claim (†) is also established. In particular, any homotopy fiber of our map

$$\mathcal{S}(M)\longrightarrow\Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(M,\nu,m)$$

is highly connected of empty. It only remains to show that the nonempty homotopy fibers correspond to elements of $\Omega^{\infty+m}\mathbf{LA}_{\bullet\%}(M,\nu,m)$ whose connected component is in the kernel of the local degree homomorphism to $8\mathbb{Z}$.

For this we use the commutative diagram

$$\begin{array}{ccccc} \pi_0\mathcal{S}(M) & \longrightarrow & \pi_m\mathbf{LA}_{\bullet\%}(M,\nu,m) & \longrightarrow & 8\mathbb{Z} \\ \downarrow{\scriptstyle\text{induced by incl.}} & & \downarrow{\scriptstyle\text{forget}} & & \downarrow{\scriptstyle=} \\ \pi_0\widetilde{\mathcal{S}}^h(M) & \longrightarrow & \pi_m\mathbf{L}_{\bullet\%}(M,\nu) & \longrightarrow & 8\mathbb{Z} \end{array}$$

where the lower row is short exact. The left-hand vertical arrow is onto by definition. Its fibers are the orbits of an action of $\mathrm{Wh}(\pi_1M)$ on $\pi_0\mathcal{S}(M)$. By direct calculation, and almost by construction, the middle vertical arrow (which is a group homomorphism) is also onto and its kernel is the image of a homomorphism

$$(2.11)\qquad \pi_0((\mathbf{A}^h{}_\%(M,\nu,m))_{h\mathbb{Z}/2})\longrightarrow\pi_m\mathbf{LA}_{\bullet\%}(M,\nu,m).$$

Here $\pi_0((\mathbf{A}^h{}_\%(M,\nu,m))_{h\mathbb{Z}/2})$ is a quotient of $\mathrm{Wh}(\pi_1M)$. Hence we need to show that the action of the Whitehead group in the upper left-hand term corresponds in the upper middle term to a translation action, using the homomorphism (2.11). This is done in chapter 13.