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Preface

This monograph consists of a series of lectures given partly at the Morningside Center of Mathematics of Chinese Academy of Sciences and the Department of Mathematics of Rutgers University, and entirely at the Department of Mathematics of the University of Franche-Comté in a course of nonlinear analysis in March and April of 2006. The material was mainly taken from some joint work with Thomas Bartsch done while the author as an Alexander von Humboldt fellow visited Giessen University. It presents some results concerning methods in critical point theory oriented towards differential equations which are variational in nature with strongly indefinite Lagrangian functionals. The author thanks greatly T. Bartsch for his kindnesses to him. He would like also to thank H. Brézis for his encouragements and F. H. Lin, Y. Y. Li for the discussions on mathematics of common interest. He also thanks L. Jeanjean for his invitation to come to Besancon and for his suggestions on the content. Finally he thanks the University of Franche-Comté for its optional support.

Yanheng Ding

Contents

Chapter 1

Introduction

The classical Calculus of Variations deals with finding minima of functionals $\Phi : X \to \mathbb{R}$ that are bounded below. The basic idea of the direct method is to consider a minimizing sequence $\Phi(u_n) \to \inf \Phi$, to find a convergent subsequence $u_{n_k} \to u$, and to show that $\Phi(u) = \inf \Phi$. In order to make this work the space X should have a topology which is rather weak for the existence of a convergent subsequence, and rather strong so that Φ is lower semicontinuous. In many applications the functional is not bounded below and instead of a minimizer one is interested in critical points. This is the concern of the Calculus of Variations in the Large or Critical Point Theory, which has undergone an enormous development in the last century due to the work of mathematicians like Morse, Lusternik, Schnirelman, Palais, Smale, Rabinowitz, Ambrosetti, Lions, Struwe, Witten, Floer and many others, with applications to problems from analysis, geometry and mathematical physics. Here one usually requires X to be a Banach manifold and Φ to be differentiable. An essential ingredient is the construction of a flow φ on X so that $\Phi(\varphi(t,u))$ is decreasing in t. This flow is used in the spirit of Morse theory, to construct deformations of sublevel sets $\Phi^c = \{u \in X : \Phi(u) \le c\}$, and to find Palais-Smale sequences $(u_n)_n$, that is: $\Phi(u_n)$ is bounded and $\Phi'(u_n) \to 0$, replacing the minimizing sequences. Typical results are the mountain pass theorem of Ambrosetti and Rabinowitz or various linking theorems. The proofs use in an essential way topological concepts based on the Brouwer or Leray-Schauder degree. The theory has also been extended to deal with (semi-)continuous functions on metric spaces, forced by problems from nonlinear elasticity (see [Degiovanni and Schuricht (1998)]). Another generalization concerns variational methods for functionals on closed convex subsets of Banach spaces developed by Struwe [Struwe (1989)] for Plateau's problem. Such functionals appear also in variational inequalities.

Motivated by several applications, for instance to finite- and infinite-dimensional Hamiltonian systems, nonlinear Schrödinger equations and nonlinear Dirac equations, we were led to consider C^1-functionals $\Phi : E = E^- \oplus E^+ \to \mathbb{R}$ defined on the product $E = E^- \oplus E^+$ of Banach spaces $E^\pm$ with $\dim E^\pm = \infty$ but where one needs to work with the weak topology on E^- in order to gain compactness. The functionals typically have the form

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u) \quad \text{for } u = u^- + u^+ \in E^- \oplus E^+. \tag{1.1}$$

Since $\dim E^\pm = \infty$ the functional is strongly indefinite. Thus all of its critical points have infinite Morse index. Moreover, $\Psi' : E \to E^*$ is not completely continuous and the Palais-Smale condition does not hold in our applications. This makes applications of Leray-Schauder degree type arguments rather subtle. On the other hand the functional $\Psi : E \to \mathbb{R}$ is weakly sequentially lower semicontinuous and $\Psi' : E \to E^*$ is weakly sequentially continuous. It turns out that the product topology

$$\mathcal{T} = (\text{weak topology on } E^-) \times (\text{norm topology on } E^+)$$

is well suited for certain arguments because $\Phi : (E, \mathcal{T}) \to \mathbb{R}$ is sequentially upper semicontinuous, and $\Phi' : (E, \mathcal{T}) \to (E^*, \text{weak}^* \text{ topology})$ is continuous. Given a finite-dimensional subspace $F \subset E^+$ the unit ball of $E^- \oplus F$ is $\mathcal{T}$-compact, and given a bounded sequence $(u_n)_n$ the negative part $(u_n^-)_n$ $\mathcal{T}$-converges (up to a subsequence). When one wants to develop critical point theory with this topology on E one needs to construct deformations on E which are $\mathcal{T}$-continuous. Deformations are usually obtained by integrating vector fields which in turn are constructed with the help of partitions of unity. So one needs to construct these in a $\mathcal{T}$-Lipschitz continuous way. A more difficult situation occurs when one is interested in "normalized solutions", that is critical points of Φ constrained to the unit sphere $SE = \{u \in E : \|u\| = 1\}$ or to other finite-codimensional submanifolds X of E.

The $\mathcal{T}$-topology on X is not metrizable, therefore the by now well developed critical point theory for (semi-)continuous functions on metric spaces cannot be applied. Instead the $\mathcal{T}$-topology is generated by a family $\mathcal{D}$ of semi-metrics. A pair $(X, \mathcal{D})$ consisting of a set X and a family of semi-metrics is called a *gage space*; see [Kelley (1995)]. The paper [Bartsch and Ding (2006I)] is a first step to develop critical point theory on gage spaces. We begin by settling some basic topological questions. We introduce the concept of a Lipschitz map $(X, \mathcal{D}) \to \mathbb{R}$ and of a Lipschitz normal gage space (disjoint closed sets can be separated by Lipschitz maps). We find conditions on $(X, \mathcal{D})$ so that X is Lipschitz normal and so that Lipschitz partitions of unity (subordinated to a given open cover) exist. In particular, we show that given a Banach space B, an arbitrary subset $B_0 \subset B$, and letting $\mathcal{D}$ be the family of semi-metrics on $X = B^*$ given by $d_b(x, y) := |\langle b, x - y\rangle_{B,B^*}|$, $b \in B_0$, the gage space $(B^*, \mathcal{D})$ is Lipschitz normal. More generally, if (Y, d_Y) is a metric space then the product gage space $(B^*, \mathcal{D}) \times (Y, d_Y)$ is Lipschitz normal and has Lipschitz partitions of unity. In addition, if B is separable and $B_0 \subset B$ is dense then also every locally closed subset (that is, an intersection of an open and a closed subset) of this product gage space is Lipschitz normal and has Lipschitz partitions of unity subordinated to an arbitrary open cover.

We then present some nonlinear problems where the abstract theory developed here can be applied. These problems arise in mechanics, physics, control theory and other topics, which are variational in nature with the feature that their solutions correspond to critical points of certain strongly indefinite functionals of the form (1.1). We are interested in the existence and multiplicity of solutions to these problems. The details are arranged in the last four chapters. In Chapter 5 we study the homoclinic orbits in the classical Hamiltonian systems

$$\begin{cases} \mathcal{J}\frac{d}{dt}z + L(t)z = R_z(t,z) & \text{for } t \in \mathbb{R} \\ z(t) \to 0 & \text{as } |t| \to \infty \end{cases}$$

with periodic or non-periodic (with respect to the time t) Hamiltonians. Chapter 6 is devoted to the standing waves of the nonlinear Schrödinger equations

$$\begin{cases} -\Delta u + V(x)u = g(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with V and g being periodic in x. We also treat here semiclassical states of a Hamiltonian system of perturbed Schrödinger equations:

$$\begin{cases} -\varepsilon^2\Delta\varphi + \alpha(x)\varphi = \beta(x)\psi + F_\psi(x,\varphi,\psi) \\ -\varepsilon^2\Delta\psi + \alpha(x)\psi = \beta(x)\varphi + F_\varphi(x,\varphi,\psi) \\ (\varphi,\psi) \in H^1(\mathbb{R}^N,\mathbb{R}^2) \end{cases}$$

without any periodicity assumption. Chapter 7 deals with localized solutions of the nonlinear Dirac equations with external fields

$$\begin{cases} -i\hbar\sum_{k=1}^{3}\alpha_k\partial_k u + \beta m u + M(x)u = G_u(x,u) & \text{for } x \in \mathbb{R}^N \\ u(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with either scale potentials (i.e., $M(x) = \beta V(x)$), or vector potentials (say, the Coulomb-type potentials). We also study semiclassical solutions (as $\hbar \to 0$). Finally, in Chapter 8 we handle solutions of homoclinic type to the systems of diffusion equations

$$\begin{cases} \partial_t u - \Delta_x u + \mathrm{b}(t,x)\cdot\nabla_x u + V(x)u = H_v(t,x,u,v) \\ -\partial_t v - \Delta_x v - \mathrm{b}(t,x)\cdot\nabla_x v + V(x)v = H_u(t,x,u,v) \end{cases}$$

for $(t,x) \in \mathbb{R}\times\mathbb{R}^N$ with $u(t,x),\ v(t,x) \to 0$ as $|t| + |x| \to \infty$. In all these problems the nonlinear terms are assumed to be either asymptotically linear or super linear. In the arguments certain analytical estimates which are needed to check the assumptions of the abstract results require different techniques. We prove new results extending the previous relative works in the literature.

Chapter 2

Lipschitz partitions of unity

Let X be a set and $\mathcal{D}$ a family of semi-metrics on X. The pair $(X, \mathcal{D})$ is called a *gage space*. We write $\mathcal{T}_d$ for the topology on X associated to the semi-metric $d : X \times X \to \mathbb{R}$. Let $\mathcal{T}_\mathcal{D}$ be the topology on X generated by all $\mathcal{T}_d$, $d \in \mathcal{D}$, that is, the coarsest topology containing all $\mathcal{T}_d$, $d \in \mathcal{D}$. If $\mathcal{D} = \{d_n : n \in \mathbb{N}\}$ is countable then $\mathcal{T}_\mathcal{D}$ is semi-metrizable. Namely, setting $\tilde{d_n} := \frac{d_n}{1+d_n}$ and $d := \sum_{n\in\mathbb{N}} \frac{1}{2^n}\tilde{d_n}$ one easily checks that $\mathcal{T}_\mathcal{D} = \mathcal{T}_d$. We call $\mathcal{D}$ saturated if $d, d' \in \mathcal{D}$ implies $\max\{d, d'\} \in \mathcal{D}$. Clearly, the family

$$\overline{\mathcal{D}} := \left\{ \max\{d_1, \ldots, d_k\} : k \in \mathbb{N},\ d_1, \ldots, d_k \in \mathcal{D} \right\}$$

is the smallest saturated family of semi-metrics on X which contains $\mathcal{D}$, the saturation of $\mathcal{D}$. It generates the same topology as $\mathcal{D}$. In this section, all topological notions refer to $\mathcal{T}_\mathcal{D} = \mathcal{T}_{\overline{\mathcal{D}}}$.

A basis of this topology is given by the sets

$$U_\varepsilon(x; d) := \{y \in X : d(x, y) < \varepsilon\}, \quad x \in X,\ d \in \overline{\mathcal{D}},\ \varepsilon > 0.$$

In fact, for $x \in X$ the sets $U_\varepsilon(x, d)$, $d \in \overline{\mathcal{D}}$, $\varepsilon > 0$, form a neighborhood basis because given semi-metrics $d_1, \ldots, d_k$, and given $\varepsilon_1, \ldots, \varepsilon_k > 0$ we set $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$, $d = \max\{d_1, \ldots, d_k\}$ and obtain

$$U_{\varepsilon_1}(x; d_1) \cap \ldots \cap U_{\varepsilon_k}(x; d_k) \supset U_\varepsilon(x; d).$$

Definition 2.1 ([Bartsch and Ding (2006I)]). *A map $f : X \to (M, d_M)$ into a semi-metric space M with semi-metric d_M is said to be Lipschitz (continuous) if there exist $d \in \overline{\mathcal{D}}$ and $\lambda > 0$ such that*

$$d_M(f(x), f(y)) \ \le \lambda d(x, y) \quad \textit{for all } x, y \in X.$$

f is called locally Lipschitz (continuous) if every $x \in X$ has a neighborhood U_x such that the restriction $f|_{U_x}$ is Lipschitz continuous.

Clearly, a (locally) Lipschitz map is continuous. Lipschitz continuity depends of course on $\mathcal{D}$ and not just on the topology $\mathcal{T}_\mathcal{D}$. We call two gage spaces $(X, \mathcal{D})$ and $(Y, \mathcal{E})$ equivalent if there exists a homeomorphism $h : X \to Y$ such that for every map $f : (Y, \mathcal{E}) \to (M, d_M)$ into a semi-metric space there holds: f is (locally)

Lipschitz if and only if $f \circ h$ is (locally) Lipschitz. In this sense, $(X, \mathcal{D})$ and $(X, \overline{\mathcal{D}})$ are equivalent.

For $Y \subset X$ and $d \in \mathcal{D}$ we set

$$d(\,.\,, Y) : X \to \mathbb{R}, \quad d(x, Y) := \inf\{d(x, y) : y \in Y\}.$$

Then

$$|d(x_1, Y) - d(x_2, Y)| \leq d(x_1, x_2),$$

so $d(\,.\,, Y)$ is Lipschitz. Clearly, the zero set of $d(\,.\,, Y)$ is the closure of Y with respect to the topology $\mathcal{T}_d$.

If $A \subset X$ is closed and $x \notin A$ then there exists a neighbourhood $U_\varepsilon(x; d) \subset X \setminus A$. The map

$$f : X \to [0, 1], \; f(y) = \min\{1, d(x, y)/\varepsilon\}$$

is Lipschitz and satisfies $f(x) = 0$, $f|_A \equiv 1$. Thus one can separate a point and a disjoint closed set by a Lipschitz map. In particular, X is completely regular. It is easy to see that one can also separate a compact set and a disjoint closed set by a Lipschitz map.

In general, X need not be normal. If X is normal we do not know whether two disjoint closed sets can be separated by a locally Lipschitz map. Similarly, if X is paracompact we do not know whether one can construct locally finite partitions of unity subordinated to an open cover of X and such that the maps in the partition of unity are locally Lipschitz. In this section we shall prove results in this direction.

Lemma 2.1. *$f : X \to M$ is locally Lipschitz if, and only if, for every $x \in X$ there exists $d \in \overline{\mathcal{D}}$, $\varepsilon > 0$, $\lambda > 0$ such that*

$$d_M(f(y), f(z)) \leq \lambda d(y, z) \quad \textit{for all } y, z \in U_\varepsilon(x; d).$$

Proof. Suppose f is locally Lipschitz. Thus there exist $d_1 \in \overline{\mathcal{D}}$, $\varepsilon > 0$ such that $f|_{U_\varepsilon(x;d_1)}$ is Lipschitz, that is, for some $d_2 \in \overline{\mathcal{D}}$, $\lambda > 0$ we have

$$d_M(f(y), f(z)) \leq \lambda d_2(y, z) \quad \text{for all } y, z \in U_\varepsilon(x; d_1).$$

Setting $d := \max\{d_1, d_2\}$ the conclusion follows. The other implication is trivial. □

Lemma 2.2. *Let $f : X \to M$ be locally Lipschitz. Then for $K \subset X$ compact there exists a neighbourhood U of K in X such that $f|_U$ is Lipschitz.*

Proof. For $x \in K$ we choose $d_x \in \overline{\mathcal{D}}$, $\varepsilon_x > 0$, $\lambda_x > 0$ such that

$$d_M(f(y), f(z)) \leq \lambda_x d_x(y, z) \quad \text{for } y, z \in U_{\varepsilon_x}(x; d_x).$$

There exist $x_1, \dots, x_n \in K$ with $K \subset \bigcup_{j=1}^n U_{\varepsilon_{x_j}/2}(x_j; d_{x_j})$. For $j = 1, \dots, n$ we set $\varepsilon_j := \varepsilon_{x_j}$, $d_j := d_{x_j}$, $\lambda_j := \lambda_{x_j}$, $U_j := U_{\varepsilon_j/2}(x_j; d_j)$, and $U := \bigcup_{j=1}^n U_j$.

We first show that $f(U)$ is bounded, that is

$$S := \sup\{d_M(f(x), f(y)) : x, y \in U\} < \infty.$$

For $x, y \in U$ there exist i, j with $x \in U_i$, $y \in U_j$. Then we have

$$\begin{aligned} d_M\big(f(x), f(y)\big) &\le d_M(f(x), f(x_i)) + d_M(f(x_i), f(x_j)) + d_M(f(x_j), f(y)) \\ &\le \lambda_i d_i(x, x_i) + d_M(f(x_i), f(x_j)) + \lambda_j d_j(x_j, y) \\ &\le \frac{\lambda_i \varepsilon_i}{2} + d_M(f(x_i), f(x_j)) + \frac{\lambda_j \varepsilon_j}{2} \\ &\le \max_{k,l} \left(\frac{\lambda_k \varepsilon_k}{2} + d_M(f(x_k), f(x_l)) + \frac{\lambda_l \varepsilon_l}{2} \right) \\ &< \infty. \end{aligned}$$

Now we prove that $f|_U$ is Lipschitz. Set $\varepsilon := \frac{1}{2}\min\{\varepsilon_1, \dots, \varepsilon_n\}$, $\lambda := \max\{\lambda_1, \dots, \lambda_n, S/\varepsilon\}$ and $d := \max\{d_1, \dots, d_n\}$. For $x, y \in U$ we choose j with $y \in U_j$. If $d_j(x, y) < \varepsilon_j/2$ then $x \in U_{\varepsilon_j}(x_j; d_j)$ and therefore

$$d_M\big(f(x), f(y)\big) \le \lambda_j d_j(x, y) \le \lambda d(x, y),$$

as required. If on the other hand $d_j(x, y) \ge \varepsilon_j/2 \ge \varepsilon$ then

$$d_M\big(f(x), f(y)\big) \le S \le \lambda d_j(x, y) \le \lambda d(x, y).$$

□

Lemma 2.3. *Let $K \subset X$ be compact and $A \subset X$ be closed such that $A \cap K = \emptyset$. Then there exists $d \in \overline{\mathcal{D}}$ with*

$$d(K, A) = \inf\{d(x, y) : x \in K, y \in A\} > 0.$$

Proof. There exist $x_1, \dots, x_n \in K$, $\varepsilon_1, \dots, \varepsilon_n > 0$ and $d_1, \dots, d_n \in \overline{\mathcal{D}}$ with $K \subset \bigcup_{j=1}^{n} U_{\varepsilon_j}(x_j; d_j)$ and $\bigcup_{j=1}^{n} U_{2\varepsilon_j}(x_j; d_j) \subset X \setminus A$. Then $d := \max\{d_1, \dots, d_n\}$ does the job: $d(K, A) \ge \min\{\varepsilon_1, \dots, \varepsilon_k\}$. □

In the situation of Lemma 2.3 the map

$$f : X \to [0, 1], \quad f(x) := \frac{d(x, K)}{d(x, K) + d(x, A)},$$

is well defined and Lipschitz, because the maps $d(., K)$, $d(., A)$ are Lipschitz and $d(x, K) + d(x, A) \ge d(K, A) > 0$ for all $x \in X$. Clearly, $f|_K \equiv 0$ and $f|_A \equiv 1$. Thus a compact set K and a disjoint closed set A can be separated by a Lipschitz map.

Definition 2.2 ([Bartsch and Ding (2006I)]). *A gage space $(X, \mathcal{D})$ is said to be Lipschitz normal if X is Hausdorff, (equivalently, $\mathcal{D}$ separates points), and if for any two closed disjoint sets $A, B \subset X$ there exists a locally Lipschitz map $f : X \to [0, 1]$ with $f|_A \equiv 0$ and $f|_B \equiv 1$.*

If $\mathcal{D} = \{d\}$ and d is a metric then $(X, \mathcal{D})$ is Lipschitz normal.

Lemma 2.4. *Suppose $(X, \mathcal{D})$ is Lipschitz normal and paracompact. Then for every open covering $\mathscr{U}$ of X there exists a subordinated locally finite partition of unity consisting of locally Lipschitz maps.*

Proof. Let $\{U_\lambda : \lambda \in \Lambda\}$ be a locally finite refinement of $\mathscr{U}$ and let $\{V_\lambda : \lambda \in \Lambda\}$ be an open cover of X with $\overline{V}_\lambda \subset U_\lambda$ for all $\lambda \in \Lambda$. Let $\rho_\lambda : X \to [0,1]$ be a locally Lipschitz map with $\rho_\lambda|_{\overline{V}_\lambda} \equiv 1$ and $\rho_\lambda|_{X\setminus U_\lambda} \equiv 0$. Then

$$\rho : X \to [1,\infty), \quad \rho(x) = \sum_{\lambda\in\Lambda} \rho_\lambda(x),$$

is well defined and locally Lipschitz because $\overline{V}_\lambda \subset \operatorname{supp}\rho_\lambda \subset \overline{U}_\lambda$, hence each $x \in X$ has a neighbourhood which intersects only finitely many $\operatorname{supp}\rho_\lambda$. The maps $\pi_\lambda := \rho_\lambda/\rho : X \to [0,1]$, $\lambda \in \Lambda$, are also locally Lipschitz and form the required partition of unity. □

We shall now find conditions on the topology of X such that $(X,\mathcal{D})$ is Lipschitz normal. Recall that X is said to be σ-compact if there exists an increasing sequence $X_1 \subset X_2 \subset \ldots$ of compact subsets of X whose union is X. If X is σ-compact then it is also paracompact (hence normal) because X is regular.

Theorem 2.1 ([Bartsch and Ding (2006I)]). *If X is σ-compact then $(X,\mathcal{D})$ is Lipschitz normal.*

Proof. Let $\emptyset = X_0 \subset X_1 \subset X_2 \subset \ldots$ be compact subsets of X with $X = \bigcup_n X_n$. Let $A, B \subset X$ be disjoint closed subsets. We construct inductively sequences $(V_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$ of open subsets of X such that $V_n \subset V_{n+1}$, $W_n \subset W_{n+1}$, $(X\setminus A) \cup (A\cap X_n) \subset V_n$, $B\cup X_n \subset W_n$, and $\overline{W}_n \cap A \subset V_n$, for all $n \in \mathbb{N}_0$. For $n = 0$ we set $V_0 := X\setminus A$ and choose a neighbourhood W_0 of B with $\overline{W}_0 \subset V_0$. If V_n and W_n have been defined for some $n \geq 0$, observe that

$$A_n := A\cap X_{n+1}\setminus V_n \subset X\setminus \overline{W}_n \quad \text{is compact.} \tag{2.1}$$

According to Lemma 2.3 there exists $d_n \in \overline{\mathcal{D}}$ with

$$\delta_n := \frac{1}{2} d_n(A_n, \overline{W}_n) > 0. \tag{2.2}$$

Now we define

$$V_{n+1} := V_n \cup U_{\delta_n}(A_n; d_n). \tag{2.3}$$

Since $(X\setminus A)\cup(A\cap X_n) \subset V_n$ we have $X_{n+1} \subset (X\setminus A)\cup(A\cap X_{n+1}) \subset V_{n+1}$. By normality there exists an open neighbourhood W'_{n+1} of X_{n+1} with $\overline{W'}_{n+1} \subset V_{n+1}$. Setting $W_{n+1} := W_n \cup W'_{n+1}$ we obtain $B\cup X_{n+1} \subset W_{n+1}$ and $\overline{W}_{n+1}\cap A \subset (\overline{W}_n \cap A)\cup\overline{W'}_{n+1} \subset V_{n+1}$. This finishes the construction of $(V_n)_{n\in\mathbb{N}_0}$ and $(W_n)_{n\in\mathbb{N}_0}$.

For $n \in \mathbb{N}_0$ we now consider the map

$$f_n : X \to [0,1], \quad f_n(x) := \frac{d_n(x, \overline{U}_{\delta_n}(A_n; d_n))}{d_n(x, \overline{U}_{\delta_n}(A_n; d_n)) + d_n(x, X\setminus U_{2\delta_n}(A_n; d_n))}.$$

This map is well defined and locally Lipschitz. Clearly we have

$$f_n(x) = 0 \quad \Leftrightarrow \quad d_n(x, A_n) \leq \delta_n$$

and

$$f_n(x) = 1 \quad \Leftrightarrow \quad d_n(x, A_n) \geq 2\delta_n.$$

Since $W_n \subset X \setminus U_{2\delta_n}(A_n; d_n)$ by (2.2) we see that $f_n|_{W_n} \equiv 1$ and therefore $f_m|_{W_n} \equiv 1$ for all $m \geq n$. This implies that the map $f := \inf_{n\in\mathbb{N}_0} f_n$ satisfies $f|_{W_n} = \min_{0\leq k\leq n} f_k|_{W_n}$. Thus f is locally Lipschitz because $\{W_n : n \in \mathbb{N}_0\}$ is an open cover of X. From $B \subset W_0 \subset W_n$ we deduce $f_n|_B \equiv 1$ for all n, so $f|_B \equiv 1$. Finally, observe that

$$V_n = (X \setminus A) \cup \bigcup_{k=0}^{n-1} U_{\delta_k}(A_k; d_k) \quad \text{for } n \geq 0,$$

hence $f|_{V_n \cap A} \equiv 0$. This yields $f|_{A\cap X_n} \equiv 0$ for all n and thus $f|_A \equiv 0$. □

It is clear that a closed subspace $Y \subset X$ with the induced family $\mathcal{D}_Y$ of semi-metrics $d|_Y : Y \times Y \to \mathbb{R}$ is Lipschitz normal when $(X, \mathcal{D})$ is Lipschitz normal. In [Smirnov (1951)] Smirnov proved that an open F_σ-subspace $Y \subset X$ of a normal space X is normal. Recall that Y is an F_σ-subspace of X if $Y = \bigcup_{n\in\mathbb{N}} Y_n$ is the union of countably many closed subsets Y_n of X. A corresponding result holds for Lipschitz normality.

Theorem 2.2 ([Bartsch and Ding (2006I)]). *Let $(X, \mathcal{D})$ be Lipschitz normal and $Y \subset X$ be an open F_σ-subspace. Then $(Y, \mathcal{D}_Y)$ is Lipschitz normal.*

Proof. Let $Y = \bigcup_{n\in\mathbb{N}} Y_n$ with $Y_n \subset X$ closed and $Y_n \subset Y_{n+1}$ for $n \in \mathbb{N}$. Consider two closed disjoint subsets A, B of Y. We write $\overline{A}$, $\overline{B}$ for the closures of A and B in X. Thus $\overline{A} \cap Y = A$, $\overline{B} \cap Y = B$ and $\overline{A} \cap \overline{B} \cap Y = \emptyset$. As in the proof of Theorem 2.1 we construct inductively open subsets V_n, W_n of Y with $V_n \subset V_{n+1}$, $W_n \subset W_{n+1}$, $(Y \setminus A) \cup (A \cap Y_n) \subset V_n$, $B \cup Y_n \subset W_n$ and $\overline{W}_n \cap A \cap Y \subset V_n$, for all $n \in \mathbb{N}_0$; here $Y_0 := \emptyset$. We set $V_0 := Y \setminus A$ and choose an open neighbourhood $W_0 \subset Y$of B such that $\overline{W}_0 \cap Y \subset V_0$. This is possible since Y is normal. Suppose V_n, W_n are defined for some $n \geq 0$. Then $A_n := A \cap Y_{n+1} \setminus V_n$ is closed in X and disjoint from the closed subset $\overline{W}_n$ of X. Since X is Lipschitz normal there exists a locally Lipschitz continuous map $f_n : X \to [0, 1]$ with $f_n|_{A_n} \equiv 0$ and $f_n|_{\overline{W}_n} \equiv 1$. We set

$$V_{n+1} := V_n \cup \{x \in Y : f_n(x) < 1/2\}$$

so that

$$Y_{n+1} \subset (Y \setminus A) \cup (A \cap Y_{n+1}) \subset V_{n+1}.$$

As a consequence of the normality of X there exists an open neighbourhood W'_{n+1} of Y_{n+1} with $\overline{W'}_{n+1} \subset V_{n+1}$. We set $W_{n+1} := W_n \cup W'_{n+1}$.

In order to define a Lipschitz map $f : Y \to [0, 1]$ which separates A and B let $\chi : [0, 1] \to [0, 1]$ be defined by $\chi(t) = 0$ for $0 \leq t \leq 1/2$, and $\chi(t) = 2t - 1$ for $1/2 \leq t \leq 1$. Now we define

$$f : Y \to [0, 1], \quad f(x) := \inf_{n\in\mathbb{N}} \chi \circ f_n(x).$$

From $f_n|_{\overline{W}_n} \equiv 1$ we deduce $f_m|_{\overline{W}_n} \equiv 1$ for all $m \geq n$, hence $f|_{\overline{W}_n} = \min_{0\leq k\leq n} \chi\circ f_k|_{\overline{W}_n}$. This implies that $f|_{\overline{W}_n}$ is locally Lipschitz for $n \in \mathbb{N}_0$, and consequently f is locally Lipschitz because $\{W_n : n \in \mathbb{N}\}$ is an open cover of Y. Moreover, $f|_B \equiv 1$ because $B \subset W_0 \subset W_n$ for all $n \in \mathbb{N}_0$. Finally, observe that

$$V_n = (Y \setminus A) \cup \bigcup_{k=0}^{n-1} \{y \in Y : f_k(y) < 1/2\},$$

and that

$$A \cap Y_n \subset A \cap V_n \subset \bigcup_{k=0}^{n-1} \{y \in Y : f_k(y) < 1/2\} \subset \bigcup_{k=0}^{n-1} \{y \in Y : \chi \circ f_k(y) = 0\}.$$

This implies $f|_{A\cap Y_n} \equiv 0$ for all $n \in \mathbb{N}$ and therefore $f|_A \equiv 0$. □

Remark 2.1. From the above proof one sees that each of the locally Lipschitz maps from Y to $[0, 1]$ of Theorem 2.2 can be required to be also a locally Lipschitz map from X to $[0, 1]$.

Next we investigate the behavior of Lipschitz normality with respect to finite products. Recall that the product $X \times Y$ of normal spaces X, Y need not be normal whereas the product of a σ-compact space X and a paracompact space Y is paracompact, hence normal by a result of Michael (see Proposition 4 of [Michael (1953)]). We extend this result to Lipschitz normality. In addition to $(X, \mathcal{D})$ we consider a set Y and a family $\mathcal{E}$ of semi-metrics on Y. Let $\mathcal{T}_\mathcal{E}$ be the associated topology on Y. For $d \in \mathcal{D}$ and $e \in \mathcal{E}$ we have an induced semi-metric $d \times e$ on $Z = X \times Y$ defined by

$$d \times e\big((x_1, y_1), (x_2, y_2)\big) := \max\{d(x_1, x_2), e(y_1, y_2)\}.$$

The topology on $X \times Y$ generated by $\mathcal{D} \times \mathcal{E} = \{d \times e : d \in \mathcal{D}, e \in \mathcal{E}\}$ is the product topology $(X, \mathcal{T}_\mathcal{D}) \times (Y, \mathcal{T}_\mathcal{E})$.

Theorem 2.3 ([Bartsch and Ding (2006I)]). *Let $(X, \mathcal{D})$ be σ-compact and $(Y, \mathcal{E})$ paracompact and Lipschitz normal. Then $(X \times Y, \mathcal{D} \times \mathcal{E})$ is Lipschitz normal.*

Proof. Let $(X_n)_{n\in\mathbb{N}}$ be an increasing sequence of compact subsets of X with $X = \bigcup_{n\in\mathbb{N}} X_n$ and $X_0 = \emptyset$. We set $Z := X \times Y$ and $Z_n := X_n \times Y$, $n \in \mathbb{N}$. Let A, B be closed subsets of Z and set $A_y := A \cap X \times \{y\}$ for $y \in Y$. We proceed as in the proof of Theorem 2.2 and construct inductively increasing sequences $(V_n)_{n\in\mathbb{N}}$, $(W_n)_{n\in\mathbb{N}}$ of open subsets of Z with $(Z \setminus A) \cup (A \cap Z_n) \subset V_n$, $B \cup Z_n \subset W_n$, $\overline{W}_n \cap A \subset V_n$. The inductive step also leads to a locally Lipschitz map $f_n : X \to [0, 1]$ which will be used later to finish the proof.

We begin with $V_0 := Z \setminus A$ and an open set W_0 satisfying $B \subset W_0$ and $\overline{W}_0 \subset V_0$. Here we used that Z is normal. Suppose V_n, W_n are given for some $n \geq 0$. Then $A_y \cap Z_{n+1} \setminus V_n$ is compact and disjoint from $\overline{W}_n$, for any $y \in Y$. Thus there exist open sets $W_y, V_y \subset X$, and $e_y \in \overline{\mathcal{E}}$, $\varepsilon_y > 0$ such that $\overline{V}_y \subset W_y$, and

$$A_y \cap Z_{n+1} \setminus V_n \subset V_y \times U_{\varepsilon_y/2}(y; e_y) \subset \overline{W}_y \times \overline{U}_{\varepsilon_y}(y; e_y) \subset Z \setminus \overline{W}_n.$$

Let $P_Y : X \times Y \to Y$ be the projection. Since X_n is compact the restriction $P_Y|_{Z_n}$ is closed. Thus $P_Y(A \cap Z_{n+1} \setminus V_n)$ is a closed subset of Y and therefore paracompact. Consequently there exists a locally finite open refinement $\{N_\lambda : \lambda \in \Lambda_n\}$ of the covering $\{U_{\varepsilon_y/2}(y; e_y) : y \in P_Y(A \cap Z_{n+1} \setminus V_n)\}$ of $P_Y(A \cap Z_{n+1} \setminus V_n)$. There also exists an open covering $\{P_\lambda : \lambda \in \Lambda_n\}$ of $P_Y(A \cap Z_{n+1} \setminus V_n)$ satisfying $\overline{P}_\lambda \subset N_\lambda$. For $\lambda \in \Lambda_n$ we choose $y_\lambda = y$ with $N_\lambda \subset U_{\varepsilon_y/2}(y; e_y)$. Then $\{V_{y_\lambda} \times P_\lambda : \lambda \in \Lambda_n\}$ and $\{W_{y_\lambda} \times N_\lambda : \lambda \in \Lambda_n\}$ are locally finite open (in $X \times Y$) covers of $A \cap Z_{n+1} \setminus V_n$ such that

$$\overline{V}_{y_\lambda} \times \overline{P}_\lambda \subset W_{y_\lambda} \times N_\lambda \subset \overline{W}_{y_\lambda} \times \overline{N}_\lambda \subset Z \setminus \overline{W}_n.$$

We set

$$V_{n+1} := V_n \cup \bigcup_{\lambda \in \Lambda_n} (V_{y_\lambda} \times P_\lambda)$$

so that

$$Z_{n+1} \subset (Z \setminus A) \cup (A \cap Z_{n+1}) \subset V_{n+1}.$$

Since $X \times Y$ is normal there exists an open neighbourhood W'_{n+1} of Z_{n+1} in $X \times Y$ with $\overline{W'}_{n+1} \subset V_{n+1}$. Setting $W_{n+1} := W_n \cup W'_{n+1}$ we clearly have $B \cup Z_{n+1} \subset W_{n+1}$ and

$$\overline{W}_{n+1} \cap A \subset (\overline{W}_n \cap A) \cup \overline{W'}_{n+1} \subset V_{n+1}.$$

Now we construct the map $f_n : X \to [0,1]$. For $\lambda \in \Lambda_n$ let $g_\lambda : X \to [0,1]$ be a locally Lipschitz map with $g_\lambda|_{\overline{V}_{y_\lambda}} \equiv 0$ and $g_\lambda|_{X \setminus W_{y_\lambda}} \equiv 1$. It exists because $(X, \mathcal{D})$ is Lipschitz normal by Theorem 2.1. Similarly, let $h_\lambda : Y \to [0,1]$ be locally Lipschitz satisfying $h_\lambda|_{\overline{P}_\lambda} \equiv 0$ and $h_\lambda|_{Y \setminus \overline{N}_\lambda} \equiv 1$. Now we define

$$f_{n+1} : X \times Y \to [0,1], \quad f_{n+1}(x,y) := \inf_{\lambda \in \Lambda_n} \max\{g_\lambda(x), h_\lambda(y)\}.$$

Setting

$$g_\lambda \times h_\lambda : X \times Y \to [0,1], \quad (x,y) \mapsto \max\{g_\lambda(x), h_\lambda(y)\},$$

we see that $g_\lambda \times h_\lambda|_{\overline{V}_{y_\lambda} \times \overline{P}_\lambda} \equiv 0$ and $g_\lambda \times h_\lambda|_{Z \setminus (W_{y_\lambda} \times N_\lambda)} \equiv 1$. Clearly $g_\lambda \times h_\lambda$ is locally Lipschitz because g_λ and h_λ have this property. Since $\{W_{y_\lambda} \times N_\lambda : \lambda \in \Lambda_n\}$ is locally finite it follows that for each $(x,y) \in X \times Y$ there exists a neighbourhood U of (x,y) and a finite set $\Lambda \subset \Lambda_n$ with $f_{n+1}|_U = \min_{\lambda \in \Lambda} g_\lambda \times h_\lambda|_U$. This implies that f_{n+1} is locally Lipschitz. Finally we define the map

$$f := \inf_n f_n : X \times Y \to [0,1], \quad f(x,y) = \inf_{n \in \mathbb{N}} f_n(x,y).$$

By construction we have $f_n|_{\overline{W}_n} \equiv 1$ because $\overline{W}_{y_\lambda} \times \overline{N}_\lambda \subset X \times Y \setminus W_n$ for every λ. This implies the local Lipschitz continuity of f as in the proof of Theorem 2.2. Clearly $f|_B \equiv 1$ because $B \subset W_0 \subset \overline{W}_n$ for every $n \in \mathbb{N}_0$. And $f|_A \equiv 0$ follows inductively from

$$A \cap Z_{n+1} \setminus V_n \subset \bigcup_{\lambda \in \Lambda_n} (V_{y_\lambda} \times P_\lambda)$$

and $f_n|_{V_{y_\lambda} \times P_\lambda} \equiv 0$ for every n. □