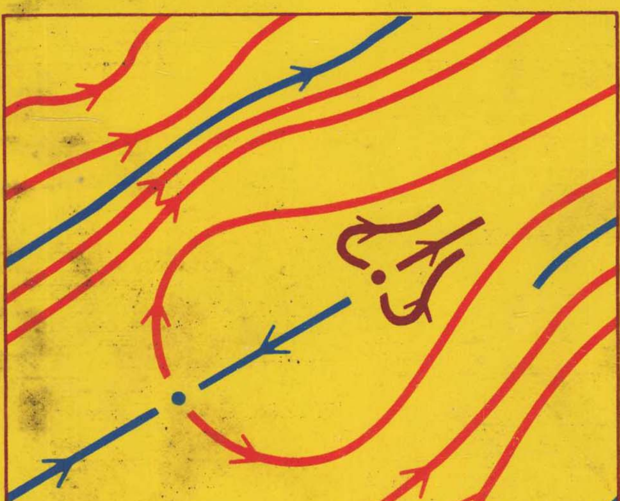
Jacob Palis, Jr.
Welington de Melo

# Geometric Theory of Dynamical Systems

## An Introduction



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## An Introduction

Translated by A. K. Manning

With 114 Illustrations

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# Geometric Theory of Dynamical Systems
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# Acknowledgments

This book grew from courses and seminars taught at IMPA and several other institutions both in Brazil and abroad, a first text being prepared for the Xth Brazilian Mathematical Colloquium. With several additions, it later became a book in the Brazilian mathematical collection *Projeto Euclides*, published in Portuguese. A number of improvements were again made for the present translation.

We are most grateful to many colleagues and students who provided us with useful suggestions and, above all, encouragement for us to present these introductory ideas on Geometric Dynamics. We are particularly thankful to Paulo Sad and, especially to Alcides Lins Neto, for writing part of a first set of notes, and to Anthony Manning for the translation into English.

# Introduction

*. . . cette étude qualitative (des équations différentielles) aura par elle-même un intérêt du premier ordre . . .*

HENRI POINCARÉ, 1881.

We present in this book a view of the Geometric Theory of Dynamical Systems, which is introductory and yet gives the reader an understanding of some of the basic ideas involved in two important topics: structural stability and genericity.

This theory has been considered by many mathematicians starting with Poincaré, Liapunov and Birkhoff. In recent years some of its general aims were established and it experienced considerable development.

More than two decades passed between two important events: the work of Andronov and Pontryagin (1937) introducing the basic concept of structural stability and the articles of Peixoto (1958–1962) proving the density of stable vector fields on surfaces. It was then that Smale enriched the theory substantially by defining as a main objective the search for generic and stable properties and by obtaining results and proposing problems of great relevance in this context. In this same period Hartman and Grobman showed that local stability is a generic property. Soon after this Kupka and Smale successfully attacked the problem for periodic orbits.

We intend to give the reader the flavour of this theory by means of many examples and by the systematic proof of the Hartman–Grobman and the Stable Manifold Theorems (Chapter 2), the Kupka–Smale Theorem (Chapter 3) and Peixoto's Theorem (Chapter 4). Several of the proofs we give are simpler than the original ones and are open to important generalizations. In Chapter 4, we also discuss basic examples of stable diffeomorphisms with infinitely many periodic orbits. We state general results on the structural stability of dynamical systems and make some brief comments on other topics, like bifurcation theory. In the Appendix to Chapter 4, we present the important concept of rotation number and apply it to describe a beautiful example of a flow due to Cherry.

Prerequisites for reading this book are only a basic course on Differential Equations and another on Differentiable Manifolds the most relevant results of which are summarized in Chapter 1. In Chapter 2 little more is required than topics in Linear Algebra and the Implicit Function Theorem and Contraction Mapping Theorem in Banach Spaces. Chapter 3 is the least elementary but certainly not the most difficult. There we make systematic use of the Transversality Theorem. Formally Chapter 4 depends on Chapter 3 since we make use of the Kupka–Smale Theorem in the more elementary special case of two-dimensional surfaces.

Many relevant results and varied lines of research arise from the theorems proved here. A brief (and incomplete) account of these results is presented in the last part of the text. We hope that this book will give the reader an initial perspective on the theory and make it easier for him to approach the literature.

*Rio de Janeiro, September 1981.*

JACOB PALIS, JR.
WELINGTON DE MELO

# List of Symbols

| | |
|---|---|
| $\mathbb{R}$ | real line |
| $\mathbb{R}^n$ | Euclidean $n$-space |
| $\mathbb{C}^n$ | complex $n$-space |
| $C^n$ | differentiability class of mappings having $n$ continuous derivatives |
| $C^\infty$ | infinitely differentiable |
| $C^\omega$ | real analytic |
| $df(p)$, $df_p$ or $Df(p)$ | derivative of $f$ at $p$ |
| $(\partial/\partial t)f$, $\partial f/\partial t$ | partial derivative |
| $D_2 f(x, y)$ | partial derivative with respect to the second variable |
| $d^n f(p)$ | $n$th derivative of $f$ at $p$ |
| $L(\mathbb{R}^n, \mathbb{R}^m)$ | space of linear mappings |
| $L^r(\mathbb{R}^m; \mathbb{R}^k)$ | space of $r$-linear mappings |
| $\\| \ \\|$ | norm |
| $g \circ f$ | composition of the mappings $g$ and $f$ |
| $\varnothing$ | empty set |
| $f \mid M$ | restriction of map $f$ to subset $M$ |
| $\overline{U}$ | closure of set $U$ |
| $TM_p$ | tangent space of $M$ at $p$ |
| $TM$ | tangent bundle of $M$ |
| $\mathfrak{X}^r(M)$ | space of $C^r$ vector fields on $M$ |
| $f_* X$ | vector field induced on the range of $f$ by $X$ |
| $X_t$ | diffeomorphism induced by flow of $X$ at time $t$ |
| $\mathcal{O}(p)$ | orbit of $p$ |
| $\omega(p)$ | $\omega$-limit set of $p$ |
| $\alpha(p)$ | $\alpha$-limit set of $p$ |
| $S^n$ | unit $n$-sphere |

| | |
|---|---|
| $T^2$ | two-dimensional torus |
| grad $f$ | gradient field of $f$ |
| $\int f$ | integral of $f$ |
| $id_M$ | identity map of $M$ |
| $\langle\ ,\ \rangle$ | Riemannian metric |
| $\langle\ ,\ \rangle_p$ | inner product in the tangent space of $p$ defined by Riemannian metric |
| $C^r(M, N)$ | space of $C^r$ mappings |
| $\\| \ \\|_r$ | $C^r$-norm |
| $\mathrm{Diff}^r(M)$ | space of $C^r$ diffeomorphisms |
| $f \mp S$ | $f$ is transversal to $S$ |
| $\mathcal{O}_X(p)$ | orbit of $X$ through $p$ |
| $\mathcal{O}_+(p)$ | positive orbit of $p$ |
| $\alpha'(t)$ | derivative at $t$ of map of interval |
| $T^n$ | $n$-dimensional torus |
| $\mathcal{L}(\mathbb{R}^n)$ | space of linear operators on $\mathbb{R}^n$ |
| $\mathcal{L}(\mathbb{C}^n)$ | complex vector space of linear operators on $\mathbb{C}^n$ |
| $L^k$ | $L \circ L \circ \cdots \circ L$ |
| $\mathrm{Exp}(L)$, $e^L$ | exponential of $L$ |
| $GL(\mathbb{R}^n)$ | group of invertible linear operators of $\mathbb{R}^n$ |
| $H(\mathbb{R}^n)$ | space of hyperbolic linear isomorphisms of $\mathbb{R}^n$ |
| $\mathcal{H}(\mathbb{R}^n)$ | space of hyperbolic linear vector fields of $\mathbb{R}^n$ |
| $\mathrm{Sp}(L)$ | spectrum of $L$ |
| $\mathcal{G}_0$ | space of vector fields having all singularities simple |
| $\det(A)$ | determinant of $A$ |
| $\mathcal{G}_1$ | space of vector fields having all singularities hyperbolic |
| $G_0$ | space of diffeomorphisms having all fixed points elementary |
| $G_1$ | space of diffeomorphisms whose fixed points are all hyperbolic |
| $C_b^0(\mathbb{R}^m)$ | space of continuous bounded maps from $\mathbb{R}^m$ to $\mathbb{R}^m$ |
| $\dim M$ | dimension of $M$ |
| $W^s(p)$ | stable manifold of $p$ |
| $W^u(p)$ | unstable manifold of $p$ |
| $W_\beta^s(p)$ | stable manifold of size $\beta$ |
| $W_\beta^u(p)$ | unstable manifold of size $\beta$ |
| $W_{\mathrm{loc}}^s(0)$ | local stable manifold |
| $W_{\mathrm{loc}}^u(0)$ | local unstable manifold |
| $\mathcal{G}_{12}$ | space of vector fields in $\mathcal{G}_1$ whose closed orbits are all hyperbolic |
| $\mathfrak{X}(T)$ | space of vector fields in $\mathcal{G}_1$ whose closed orbits of period $\leq T$ are all hyperbolic |
| $L_\alpha(X)$ | union of the $\alpha$-limit sets of orbits of $X$ |
| $L_\omega(X)$ | union of the $\omega$-limit sets of orbits of $X$ |
| $\Omega(X)$ | set of nonwandering points of $X$ |
| M–S | set of Morse–Smale vector fields |
| $\partial M$ | boundary of $M$ |
| int $A$ | interior of set $A$ |

# Contents

Chapter 1

# Differentiable Manifolds and Vector Fields

This chapter establishes the concepts and basic facts needed for understanding later chapters.

First we set out some classical results from Calculus in $\mathbb{R}^n$, Ordinary Differential Equations and Submanifolds of $\mathbb{R}^n$. Next we define vector fields on manifolds and we apply the local results of the Theory of Differential Equations in $\mathbb{R}^n$ to this case. We introduce the qualitative study of vector fields, with the concepts of $\alpha$- and $\omega$-limit sets, and prove the important Poincaré–Bendixson Theorem.

In Section 2 we define the $C^r$ topology on the set of differentiable maps between manifolds. We show that the set of $C^r$ maps with the $C^r$ topology is a separable Baire space and that the $C^\infty$ maps are dense in it. From this we obtain topologies with the same properties for the spaces of vector fields and diffeomorphisms.

Section 3 is devoted to the Transversality Theorem, which we shall use frequently.

We conclude the chapter by establishing the general aims of the Geometric or Qualitative Theory of Dynamical Systems. In particular we discuss the concepts of topological equivalence and structural stability for differential equations defined on submanifolds of $\mathbb{R}^n$.

## §0 Calculus in $\mathbb{R}^n$ and Differentiable Manifolds

In this section we shall state some concepts and basic results from Calculus in $\mathbb{R}^n$, Differential Equations and Differentiable Manifolds. The proofs of the facts set out here on Calculus in $\mathbb{R}^n$ can be found in [46], [48]; on Differential

Equations in the much recommended introductory texts [4], [41], [116] or the more advanced ones [33], [35] and also [47]; on Differentiable Manifolds in [29], [38], [49].

Let $f: U \subset \mathbb{R}^m \to \mathbb{R}^k$ be a map defined on the open subset $U$ of $\mathbb{R}^m$. We say that $f$ is *differentiable at a point $p$ of $U$* if there exists a linear transformation $T: \mathbb{R}^m \to \mathbb{R}^k$ such that, for small $v$, $f(p+v) = f(p) + T(v) + R(v)$ with $\lim_{v\to 0} R(v)/\|v\| = 0$. We say that the linear map $T$ is the *derivative of $f$ at $p$* and write it as $df(p)$ or sometimes $df_p$ or $Df(p)$. The existence of the derivative of $f$ at $p$ implies, in particular, that $f$ is continuous at $p$. If $f$ is differentiable at each point of $U$ we have a map $df: L(\mathbb{R}^m, \mathbb{R}^k)$ which to each $p$ in $U$ associates the derivative of $f$ at $p$. Here $L(\mathbb{R}^m, \mathbb{R}^k)$ denotes the vector space of linear maps from $\mathbb{R}^m$ to $\mathbb{R}^k$ with the norm $\|T\| = \sup\{\|Tv\|;\ \|v\| = 1\}$. If $df$ is continuous we say that $f$ is *of class $C^1$* in $U$. It is well known that $f$ is $C^1$ if and only if the partial derivatives of the coordinate functions of $f$, $\partial f^i/\partial x_j: U \to \mathbb{R}$, exist and are continuous. The matrix of $df(p)$ with respect to the canonical bases of $\mathbb{R}^m$ and $\mathbb{R}^k$ is $[(\partial f^i/\partial x_j)(p)]$. Analogously we define $d^2f(p)$ as the derivative of $df$ at $p$. Thus $d^2f(p)$ belongs to the space $L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^k))$, which is isomorphic to the space $L^2(\mathbb{R}^m; \mathbb{R}^k)$ of bilinear maps from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^k$. The norm induced on $L^2(\mathbb{R}^m; \mathbb{R}^k)$ by this isomorphism is $\|B\| = \sup\{\|B(u,v)\|;\ \|u\| = \|v\| = 1\}$. We say that $f$ is *of class $C^2$* in $U$ if $d^2f: U \to L^2(\mathbb{R}^m; \mathbb{R}^k)$ is continuous. By induction we define $d^rf(p)$ as the derivative at $p$ of $d^{r-1}f$. We have $d^rf(p) \in L^r(\mathbb{R}^m; \mathbb{R}^k)$, where $L^r(\mathbb{R}^m; \mathbb{R}^k)$ is the space of $r$-linear maps with the norm $\|C\| = \sup\{\|C(v_1, \ldots, v_r)\|;\ \|v_1\| = \cdots = \|v_r\| = 1\}$. Then we say that $f$ is *of class $C^r$* in $U$ if $d^rf: U \to L^r(\mathbb{R}^m; \mathbb{R}^k)$ is continuous. Finally, $f$ is *of class $C^\infty$* in $U$ if it is of class $C^r$ for all $r \geq 0$. We remark that $f$ is of class $C^r$ if and only if all the partial derivatives up to order $r$ of the coordinate functions of $f$ exist and are continuous. Let $U$, $V$ be open sets in $\mathbb{R}^m$ and $f: U \to V$ a surjective map of class $C^r$. We say that $f$ is a *diffeomorphism of class $C^r$* if there exists a map $g: V \to U$ of class $C^r$ such that $g \circ f$ is the identity on $U$.

**0.0 Proposition.** *Let $U \subset \mathbb{R}^m$ be an open set and $f_n: U \to \mathbb{R}^k$ be a sequence of maps of class $C^1$. Suppose that $f_n$ converges pointwise to $f: U \to \mathbb{R}^k$ and that the sequence $df_n$ converges uniformly to $g: U \to L(\mathbb{R}^m, \mathbb{R}^k)$. Then $f$ is of class $C^1$ and $df = g$.* □

**0.1 Proposition** (Chain Rule). *Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open sets. If $f: U \to \mathbb{R}^n$ is differentiable at $p \in U$, $f(U) \subset V$ and $g: V \to \mathbb{R}^k$ is differentiable at $q = f(p)$, then $g \circ f: U \to \mathbb{R}^k$ is differentiable at $p$ and $d(g \circ f)(p) = dg(f(p)) \circ df(p)$.* □

**Corollary 1.** *If $f$ and $g$ are both of class $C^r$, then $g \circ f$ is of class $C^r$.* □

**Corollary 2.** *If $f: U \to \mathbb{R}^k$ is differentiable at $p \in U$ and $\alpha: (-1, 1) \to U$ is a curve such that $\alpha(0) = p$ and $(d/dt)\alpha(0) = v$, then $f \circ \alpha$ is a curve which is differentiable at $0$ and $(d/dt)(f \circ \alpha)(0) = df(p)v$.* □

**0.2 Theorem** (Inverse Function). *Let $f: U \subset \mathbb{R}^m \to \mathbb{R}^m$ be of class $C^r$, $r \geq 1$. If $df(p): \mathbb{R}^m \to \mathbb{R}^m$ is an isomorphism, then $f$ is a local diffeomorphism at $p \in U$ of class $C^r$; that is, there exist neighbourhoods $V \subset U$ of $p$ and $W \subset \mathbb{R}^m$ of $f(p)$ and a $C^r$ map $g: W \to V$ such that $g \circ f = I_V$ and $f \circ g = I_W$, where $I_V$ denotes the identity map of $V$ and $I_W$ the identity of $W$.* □

**0.3 Theorem** (Implicit Function). *Let $U \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open set and $f: U \to \mathbb{R}^n$ a $C^r$ map, $r \geq 1$. Let $z_0 = (x_0, y_0) \in U$ and $c = f(z_0)$. Suppose that the partial derivative with respect to the second variable, $D_2 f(z_0)$: $\mathbb{R}^n \to \mathbb{R}^n$, is an isomorphism. Then there exist open sets $V \subset \mathbb{R}^m$ containing $x_0$ and $W \subset U$ containing $z_0$ such that, for each $x \in V$, there exists a unique $\xi(x) \in \mathbb{R}^n$ with $(x, \xi(x)) \in W$ and $f(x, \xi(x)) = c$. The map $\xi: V \to \mathbb{R}^n$, defined in this way, is of class $C^r$ and its derivative is given by $d\xi(x) = [D_2 f(x, \xi(x))]^{-1} \circ D_1 f(x, \xi(x))$.* □

**Remark.** These theorems are also valid in Banach spaces.

**0.4 Theorem** (Local Form for Immersions). *Let $U \subset \mathbb{R}^m$ be open and $f: U \to \mathbb{R}^{m+n}$ a $C^r$ map, $r \geq 1$. Suppose that, for some $x_0 \in U$, the derivative $df(x_0): \mathbb{R}^m \to \mathbb{R}^{m+n}$ is injective. Then there exist neighbourhoods $V \subset U$ of $x_0$, $W \subset \mathbb{R}^n$ of the origin and $Z \subset \mathbb{R}^{m+n}$ of $f(x_0)$ and a $C^r$ diffeomorphism $h$: $Z \to V \times W$ such that $h \circ f(x) = (x, 0)$ for all $x \in V$.* □

**0.5 Theorem** (Local Form for Submersions). *Let $U \subset \mathbb{R}^{m+n}$ be open and $f: U \to \mathbb{R}^n$ a $C^r$ map, $r \geq 1$. Suppose that, for some $z_0 \in U$, the derivative $df(z_0)$ is surjective. Then there exist neighbourhoods $Z \subset U$ of $z_0$, $W \subset \mathbb{R}^n$ of $c = f(z_0)$ and $V \subset \mathbb{R}^m$ of the origin and a $C^r$ diffeomorphism $h$: $V \times W \to Z$ such that $f \circ h(x, w) = w$ for all $x \in V$ and $w \in W$.* □

Let $f: U \subset \mathbb{R}^m \to \mathbb{R}^n$ be a $C^r$ map, $r \geq 1$. A point $x \in U$ is *regular* if $df(x)$ is surjective; otherwise $x$ is a *critical point*. A point $c \in \mathbb{R}^n$ is a *regular value* if every $x \in f^{-1}(c)$ is a regular point; otherwise $c$ is a *critical value*. A subset of $\mathbb{R}^n$ is *residual* if it contains a countable intersection of open dense subsets. By Baire's Theorem every residual subset of $\mathbb{R}^n$ is dense.

**0.6 Theorem** (Sard [64]). *If $f: U \subset \mathbb{R}^m \to \mathbb{R}^n$ is of class $C^\infty$ then the set of regular values of $f$ is residual in $\mathbb{R}^n$.* □

We should remark here that if $f^{-1}(c) = \varnothing$ then $c$ is a regular value. For the existence of a regular point $x \in U$ we need $m \geq n$. If $m < n$ all the points of $U$ are critical and, therefore, $f(U)$ is "meagre" in $\mathbb{R}^n$, that is $\mathbb{R}^n - f(U)$ is residual.

We are going now to state some basic results on differential equations. A *vector field* on an open set $U \subset \mathbb{R}^m$ is a map $X: U \to \mathbb{R}^m$. We shall only consider fields of class $C^r$, $r \geq 1$. An *integral curve* of $X$, through a point

$p \in U$, is a differentiable map $\alpha: I \to U$, where $I$ is an open interval containing 0, such that $\alpha(0) = p$ and $\alpha'(t) = X(\alpha(t))$ for all $t \in I$. We say that $\alpha$ is a *solution* of the differential equation $dx/dt = X(x)$ with initial condition $x(0) = p$.

**0.7 Theorem** (Existence and Uniqueness of Solutions). *Let $X$ be a vector field of class $C^r$, $r \geq 1$, on an open set $U \subset \mathbb{R}^m$ and let $p \in U$. Then there exists an integral curve of $X$, $\alpha: I \to U$, with $\alpha(0) = p$. If $\beta: J \to U$ is another integral curve of $X$ with $\beta(0) = p$ then $\alpha(t) = \beta(t)$ for all $t \in I \cap J$.* □

A *local flow* of $X$ at $p \in U$ is a map $\varphi: (-\varepsilon, \varepsilon) \times V_p \to U$, where $V_p$ is a neighbourhood of $p$ in $U$, such that, for each $q \in V_p$, the map $\varphi_q: (-\varepsilon, \varepsilon) \to U$, defined by $\varphi_q(t) = \varphi(t, q)$, is an integral curve through $q$; that is, $\varphi(0, q) = q$ and $(\partial/\partial t)\varphi(t, q) = X(\varphi(t, q))$ for all $(t, q) \in (-\varepsilon, \varepsilon) \times V_p$.

**0.8 Theorem.** *Let $X$ be a vector field of class $C^r$ in $U$, $r \geq 1$. For all $p \in U$ there exists a local flow, $\varphi: (-\varepsilon, \varepsilon) \times V_p \to U$, which is of class $C^r$. We also have*

$$D_1 D_2 \varphi(t, q) = DX(\varphi(t, q)) \cdot D_2 \varphi(t, q)$$

*and $D_2 \varphi(0, q)$ is the identity map of $\mathbb{R}^m$, where $D_1$ and $D_2$ denote the partial derivatives with respect to the first and second variables.* □

We can also consider vector fields that depend on a parameter and the dependence of their solutions on the parameter. Let $E$ be a Banach space and $F: E \times U \to \mathbb{R}^m$ a $C^r$ map, $r \geq 1$. For each $e \in E$ the map $F_e: U \to \mathbb{R}^m$, defined by $F_e(p) = F(e, p)$, is a vector field on $U$ of class $C^r$. The following theorem shows that the solutions of this field $F_e$ depend continuously on the parameter $e \in E$.

**0.9 Theorem.** *For every $e \in E$ and $p \in U$ there exist neighbourhoods $W$ of $e$ in $E$ and $V$ of $p$ in $U$ and a $C^r$ map $\varphi: (-\varepsilon, \varepsilon) \times V \times W \to U$ such that*

$$\varphi(0, q, \lambda) = q,$$
$$D_1 \varphi(t, q, \lambda) = F(\lambda, \varphi(t, q, \lambda))$$

*for every $(t, q, \lambda) \in (-\varepsilon, \varepsilon) \times V \times W$.* □

Next we introduce the concept of differentiable manifold. To simplify the exposition we define manifolds as subsets of $\mathbb{R}^k$. At the end of this section we discuss the abstract definition.

Let $M$ be a subset of Euclidean space $\mathbb{R}^k$. We shall use the induced topology on $M$; that is, $A \subset M$ is open if there exists an open set $A' \subset \mathbb{R}^k$ such that $A = A' \cap M$. We say that $M \subset \mathbb{R}^k$ is a *differentiable manifold* of dimension $m$ if, for each point $p \in M$, there exists a neighbourhood $U \subset M$ of $p$ and a homeomorphism $x: U \to U_0$, where $U_0$ is an open subset of $\mathbb{R}^m$, such that

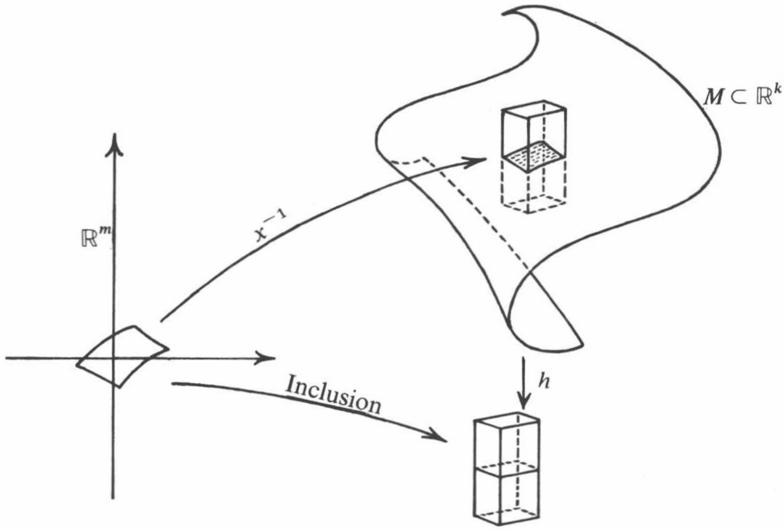


Figure 1

the inverse homeomorphism $x^{-1}: U_0 \to U \subset \mathbb{R}^k$ is an immersion of class $C^\infty$. That is, for each $u \in U_0$, the derivative $dx^{-1}(u): \mathbb{R}^m \to \mathbb{R}^k$ is injective. In this case we say that $(x, U)$ is a *local chart around* $p$ and $U$ is a *coordinate neighbourhood* of $p$. If the homeomorphisms $x^{-1}$ above are of class $C^r$ we say that $M$ is a manifold of class $C^r$. What we have called a differentiable manifold corresponds to one of class $C^\infty$. It follows from the Local Form for Immersions 0.4 that, if $(x, U)$ is a local chart around $p \in M$, then there exist neighbourhoods $A$ of $p$ in $\mathbb{R}^k$, $V$ of $x(p)$ and $W$ of the origin in $\mathbb{R}^{k-m}$ and a $C^\infty$ diffeomorphism $h: A \to V \times W$ such that, for all $q \in A \cap M$, we have $h(q) = (x(q), 0)$. In particular, a local chart is the restriction of a $C^\infty$ map of an open subset of $\mathbb{R}^k$ into $\mathbb{R}^m$ (Figure 1). From this remark we obtain the following proposition.

**0.10 Proposition.** *Let* $x: U \to \mathbb{R}^m$ *and* $y: V \to \mathbb{R}^m$ *be local charts in* $M$. *If* $U \cap V \neq \varnothing$ *then the change of coordinates* $y \circ x^{-1}: x(U \cap V) \to y(U \cap V)$ *is a* $C^\infty$ *diffeomorphism* (Figure 2). □

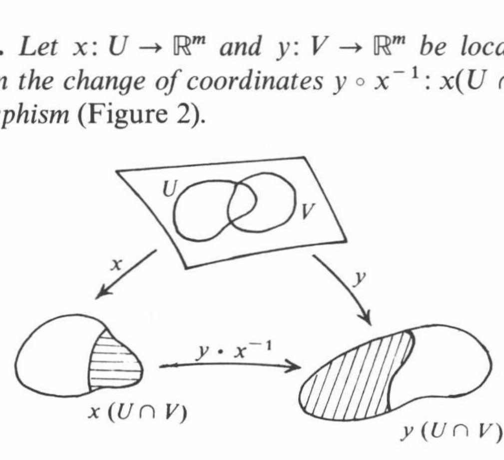


Figure 2

We shall now define differentiable maps between manifolds. Let $M^m$ and $N^n$ be manifolds and $f: M^m \to N^n$ a map. We say that $f$ is *of class* $C^r$ if, for each point $p \in M$, there exist local charts $x: U \to \mathbb{R}^m$ around $p$ and $y: V \to \mathbb{R}^n$ with $f(U) \subset V$ such that $y \circ f \circ x^{-1}: x(U) \to y(V)$ is of class $C^r$. As the changes of coordinates are of class $C^\infty$ this definition is independent of the choice of charts.

Let us consider a differentiable curve $\alpha: (-\varepsilon, \varepsilon) \to M \subset \mathbb{R}^k$ with $\alpha(0) = p$. It is easy to see that $\alpha$ is differentiable according to the above definition if and only if $\alpha$ is differentiable as a curve in $\mathbb{R}^k$. Hence there exists a tangent vector $(d\alpha/dt)(0) = \alpha'(0)$. The set of vectors tangent to all such curves $\alpha$ is called the tangent space to $M$ at $p$ and denoted by $TM_p$. Let us consider a local chart $x: U \to \mathbb{R}^m$, $x(p) = 0$. It is easy to see that the image of the derivative $dx^{-1}(0)$ coincides with $TM_p$. Thus $TM_p$ is a vector space of dimension $m$.

Let $f: M \to N$ be a differentiable map and $v \in TM_p$, $p \in M$. Consider a differentiable curve $\alpha: (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$ and $\alpha'(0) = v$. Then $f \circ \alpha: (-\varepsilon, \varepsilon) \to N$ is a differentiable curve, so we can define $df(p)v = (d/dt)(f \circ \alpha)(0)$. This definition is independent of the curve $\alpha$.

The map $df(p): TM_p \to TN_{f(p)}$ is linear and is called the *derivative of $f$ at $p$*.

As a differentiable manifold is locally an open subset of a Euclidean space all the theorems from Calculus that we listed earlier extend to manifolds.

**0.11 Proposition** (Chain Rule). *Let $f: M \to N$ and $g: N \to P$ be maps of class $C^r$ between differentiable manifolds. Then $g \circ f: M \to P$ is of class $C^r$ and $d(g \circ f)(p) = dg(f(p)) \circ df(p)$.* □

A map $f: M \to N$ is a $C^r$ *diffeomorphism* if it is of class $C^r$ and has an inverse $f^{-1}$ of the same class. In this case, for each $p \in M$, $df(p): TM_p \to TN_{f(p)}$ is an isomorphism whose inverse is $df^{-1}(f(p))$. In particular, $M$ and $N$ have the same dimension. We say that $f: M \to N$ is a *local diffeomorphism* at $p \in M$ if there exist neighbourhoods $U(p) \subset M$ and $V(f(p)) \subset N$ such that the restriction of $f$ to $U$ is a diffeomorphism onto $V$.

**0.12 Proposition** (Inverse Function). *If $f: M \to N$ is of class $C^r$, $r \geq 1$, and $df(p)$ is an isomorphism for some $p \in M$ then $f$ is a local diffeomorphism of class $C^r$ at $p$.* □

Now consider a subset $S$ of a manifold $M$. $S$ is a *submanifold of class* $C^r$ of $M$ of dimension $s$ if, for each $p \in S$, there exist open sets $U \subset M$ containing $p$, $V \subset \mathbb{R}^s$ containing 0 and $W \subset \mathbb{R}^{m-s}$ containing 0 and a diffeomorphism of class $C^r$ $\varphi: U \to V \times W$ such that $\varphi(S \cap U) = V \times \{0\}$ (Figure 3).

We remark that $\mathbb{R}^k$ is a differentiable manifold and that, if $M \subset \mathbb{R}^k$ is a manifold as defined above, then $M$ is a submanifold of $\mathbb{R}^k$. The submanifolds of $M \subset \mathbb{R}^k$ are those submanifolds of $\mathbb{R}^k$ that are contained in $M$.