SOCIAL CONSTRUCTIVISM AS A PHILOSOPHY OF MATHEMATICS

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Introduction

Mathematics is one of the great cultural achievements of humankind. Every schooled person understands the rudiments of number and measures and sees the world through this quantifying conceptual framework. By these means mathematics provides the language of the socially all-important practices of work, commerce, and economics. In addition, digital computers and the full range of information technology applications are all regulated by and speak to each other exclusively in the language of mathematics, and they would not be possible without it. Thus mathematics is essential to the modern technological way of life and the social outlook that accompanies it.

In contrast, some of the deepest and most abstract speculations of the human mind concern the nature and relations of objects found only in the virtual reality of mathematics. Infinities, paradoxes, logical deduction, perfect harmonies, structures and symmetries, and many other concepts are all analyzed and explored definitively in mathematics. Thus mathematics provides the language of daring abstract thought. Related to this, mathematics is the language of certainty. For over two thousand years thinkers have regarded mathematics as the only self-subsistent area of thought that provides certainty, necessity, and absolute universal truth. So mathematics might be said to have, in addition to a mundane utilitarian role, an epistemological role, an ideological role, and even a mystical role in human culture

Despite being partly familiar to all, because of these contradictory aspects, mathematics remains an enigma and a mystery at the heart of human culture. It is both the language of the everyday world of commercial life and that of an unseen and perfect virtual reality. It includes both free-ranging ethereal speculation and rock-hard certainty. How can this mystery be explained? How can it be unraveled? The philosophy of mathematics is meant to cast some light on this mystery: to explain the nature and character of mathematics. However this philosophy can be purely technical, a product of the academic love of technique expressed in the foundations of mathematics or in philosophical virtuosity. Too often the outcome of philosophical inquiry is to provide detailed answers to the *how* questions of mathematical certainty and existence, taking for granted the received ideology of mathematics, but with too little attention to the deeper *why* questions. Thus, for example, there are still real controversies in the philosophy of mathematics over whether the history of mathematics has any bearing on its philosophy, and whether the experiences and practices of working mathematicians can shed any light on questions of mathematical knowledge. In the philosophy of science such questions have long been settled affirmatively. But this is not yet the case in the philosophy of mathematics. One of my goals in writing this book is to try to lift the veil and to demystify mathematics; to show that for all its wonder it remains a set of human practices, grounded, like everything else, in the material world we inhabit.

In the philosophy of mathematics a number of voices have been heard calling for a more naturalistic account of mathematics. In differing ways Davis and Hersh (1980), Kitcher (1984), Lakatos (1976), Tymoczko (1986a), Tiles (1991), Wittgenstein (1956), and others have argued for a critical reexamination of traditional presuppositions about the certainty of mathematical knowledge. Kitcher and Aspray (1988) suggest that these voices make up a new "maverick" tradition in the philosophy of mathematics which is concerned to accommodate current and past mathematical practices in a philosophical account of mathematics.

Outside of the philosophy of mathematics there has been more progress. First of all, a number of different traditions of thought in sociology, psychology, history and philosophy have been drawing on the central idea of the social construction of knowledge as a way of accounting for science and mathematics naturalistically. Second, a growing number of researchers have been drawing on other disciplines to account for the nature of mathematics, including Bloor (1976), Livingston (1986) and Restivo (1992), from sociology; Ascher (1991), D'Ambrosio (1985), Wilder (1981) and Zaslavsky (1973) from cultural studies and ethnomathematics; Rotman (1987, 1993) from semiotics, Aspray and Kitcher (1988), Joseph (1991) and Gillies (1992) from the history of mathematics, and Bishop (1988), Ernest (1991) and Skovsmose (1994) from education.

This book can be located at the intersection of these traditions. It draws its central explanatory scheme from the interdisciplinary social constructionist approaches currently burgeoning in the human sciences. It gains confidence from the parallels in multidisciplinary and multidimensional accounts of mathematics. But it draws its central concepts and inspiration from the emerging maverick tradition in the philosophy of mathematics.

The book begins with a strong critique of absolutist views of mathematical knowledge in the philosophy of mathematics (chap. 1) and traditional approaches to the philosophy of mathematics in general (chap. 2). It argues that the philosophy of mathematics needs to be reconceptualized and broadened to accommodate the social and historical factors mentioned above.

In the next part, the philosophies of Wittgenstein (chap. 3) and Lakatos (chap. 4) are critically reviewed and then used as a basis for an account of the social construction of mathematical knowledge (chap. 5). This involves redefining the concept of mathematical knowledge to include tacit and shared components, as well as developing an account of the "conversational" mechanism for the social genesis and justification of mathematical knowledge. This is the generalized logic of mathematical discovery, extending Lakatos's heuristic.

Chapter 6 develops the central idea of conversation which underpins social constructivism. This requires breaking new ground in exploring the textual basis of mathematical knowledge and the rhetorical functions of mathematical language and proof. The role of conversation in the formation of mind and in social construction of subjective knowledge of mathematics is also developed (chap. 7), together with the role of semiotic tools and rhetoric in the learning of mathematics. A surprising analogy is revealed between the social genesis and justification of "objective" mathematical knowledge, on the one hand, and that of subjective mathematical knowledge, on the other. It is argued that the philosophy of mathematics must consider the social construction of the individual mathematician and her/his creativity, if it is to account for mathematical knowledge naturalistically.

The book concludes by evaluating its proposals in the light of its critique of the philosophy of mathematics and argues that, contrary to traditional perceptions, a socially constructed mathematics has a vital social responsibility to bear (chap. 8).

Followers of my work will know that I have been working on social constructivism for more than a decade, and it will come as no surprise that this account builds on an earlier version (Ernest 1991). The greatest similarities between the two versions occur in chapters 1 and 2 of this book, where I felt it was necessary to go over and improve the arguments against absolutism in the philosophy of mathematics and for the reconceptualization of the field. In addition to the goal of making the argument self-contained, there is enough novelty in these chapters to justify including them in their own right, even for seasoned readers of the earlier work. For example, there is a new argument that a reconceptualized philosophy of mathematics should offer an account of the learning of mathematics and its role in the onward transmission of mathematical knowledge.

The present work is not merely an extension and elaboration of the earlier version of social constructivism in Ernest (1991). In addition to being almost three times the length there are a number of significant conceptual differences between this and the earlier version, including the following improvements:

1. Deeper analyses of Wittgenstein's and Lakatos's thought
2. Less reliance on language as an explicit foundation of subjective knowledge of mathematics, with more emphasis on tacit knowledge, and on language and rhetoric in accounting for "objective" mathematical knowledge
3. Recognition of the semiotic basis of mathematics and mathematical knowledge
4. A shift from a Piagetian/constructivist view of mind to a social view based on Mead, Vygotsky, and others (see also Ernest 1994b)
5. Greater recognition of the culture-boundedness of all knowledge, and the necessity of identifying its material basis
6. A diminished concern to maintain the boundaries between history, sociology, psychology and the philosophy of mathematics

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CHAPTER 1

A Critique of Absolutism in the Philosophy of Mathematics

Historically, mathematics has long been viewed as the paradigm of infallibly secure knowledge. Euclid and his colleagues first constructed a magnificent logical structure around 2,300 years ago in the *Elements*, which at least until the end of the nineteenth century was taken as the paradigm for establishing incorrigible truth. Descartes ([1637] 1955) modeled his epistemology directly on the method and style of geometry. Hobbes claimed that "geometry . . . is the only science . . . bestow[ed] on [hu]mankind" (Hobbes [1651] 1962, 77). Newton in his *Principia* and Spinoza in his *Ethics* used the form of the *Elements* to strengthen their claims of systematically expounding the truth.[1] This logical form reached its ultimate expression in *Principia Mathematica*, in which Whitehead and Russell (1910–13) reapplied it to mathematics, while paying homage to Newton with their title. As part of the logicist program, *Principia Mathematica* was intended to provide a rigorous and certain foundation for all of mathematical knowledge. Thus mathematics has long been taken as the source of the most infallible knowledge known to humankind, and much of this is due to the logical structure of its presentation and justification.

With this background, a philosophical inquiry into mathematics raises questions including: What is the basis for mathematical knowledge? What is the nature of mathematical truth? What characterizes the truths of mathematics? What is the justification for their assertion? Why are the truths of mathematics necessary truths? How absolute is this necessity?

The Nature of Knowledge

The question, What is knowledge? lies at the heart of philosophy, and mathematical knowledge plays a special part. The standard philosophical answer, which goes back to Plato, is that knowledge is justified true belief. To put it differently, propositional knowledge consists of propositions which are accepted (i.e., believed), provided there are adequate grounds fully available to the believer for asserting them (Sheffler 1965; Chisholm 1966; Woozley 1949). This way of putting it avoids presupposing the truth of what is known, although traditional accounts require it, by referring instead to adequate grounds, which also include the justificatory element. The phrase "fully available" circumvents the difficulty caused when the adequate grounds exist but are not in the cognizance of the believer.[2]

Knowledge is classified on the basis of the grounds for its assertion. *A priori* knowledge consists of propositions which are asserted on the basis of reason alone, without recourse to observations of the world. Here reason consists of the use of deductive logic and the meanings of terms, typically to be found in definitions. In contrast, empirical or *a posteriori* knowledge consists of propositions asserted on the basis of experience, that is, based on observations of the world (Woozley 1949). This basis refers strictly to the empirical *justificatory* basis of a posteriori knowledge, not its genesis. Indeed, such knowledge may be initially generated by pure thought, whilst a priori knowledge, such as that of mathematics, may be first generated by induction from empirical observation. Such origins are immaterial; only the grounds for asserting the knowledge matter. This distinction is first to be found in Kant ([1781] 1961), but also occurs implicitly in earlier work, such as in Leibniz ("truths of reason" versus "truths of fact") and Hume ("matters of fact" versus "matters of reason"), Vico ("verum" or a priori truth versus "certum" or the empirical), as well as being anticipated by Plato.

Kant not only distinguishes a priori and a posteriori knowledge, on the basis of the means of verification used to justify them, but also distinguishes between *analytic* and *synthetic* propositions. A proposition is *analytic* if it follows from the law of contradiction, that is, if its denial is logically inconsistent.[3] Kant argued that mathematical knowledge is synthetic *a priori,* since it is based on reason, not empirical facts, but does not follow from the law of contradiction alone. The standard view in epistemology (see Feigl and Sellars 1949, for example) is that Kant was wrong and mathematics is analytic, and that the analytic can be identified with the *a priori* and the synthetic with the *a posteriori.* According to this view, mathematical theorems add nothing to knowledge which is not implicitly contained in the premises *logically,* although *psychologically* the theorems may be novel.

The debate is not straightforward, for a number of reasons. First of all, Kant believed in a universal logic, whereas now we recognize alternative systems in logic (Haack 1974, 1978). He also believed that mathematical theories such as Euclidean geometry and arithmetic are the necessary logical outcomes of reason. (Non-Euclidean geometry and nonstandard arithmetics were simply not possible in his system.) He concluded that although the truths of mathematics are necessary, they do not follow from the law of contradiction, but from the forms that human understanding takes, by its very nature.

A number of modern philosophers have agreed with Kant, at least so far as to dissent from the received view that identifies the analytic with the *a priori* and the synthetic with the *a posteriori*. Hintikka (1973) argues that some mathematical proofs require the addition of auxiliary elements or concepts, and hence add something unforeseen and logically novel to the mathematical knowledge. Since such proofs do not rest on the law of contradiction alone, he argues that they are synthetic, in both senses, as well as *a priori*. Brouwer and Wittgenstein (as I shall show below and in chap. 3, respectively) similarly accept that some mathematical knowledge is both synthetic and *a priori*. Finally, some others, such as Quine (1953b, 1970) and White argue that "the analytic and the synthetic [is] an untenable dualism" (White 1950). Their view is that the boundary between the two classes cannot be fixed determinately. Quine (1960) goes on to elaborate his view that mathematical and empirical scientific knowledge cannot be neatly partitioned into the analytic and synthetic. He argues that the whole of language is a "vast verbal structure," and it is not possible to separate out those parts which have empirical import from those that do not; "this structure of interconnected sentences is a single connected fabric including all sciences, and . . . logical truths" (Quine 1960, 12).

These subtleties and dissenting views notwithstanding, according to the received view mathematical knowledge is classified as *a priori* knowledge, since it consists of propositions asserted on the basis of reason alone. Reason includes deductive logic and definitions which are used, in conjunction with an assumed set of mathematical axioms or postulates, as a basis from which to infer mathematical knowledge. Thus the foundation of mathematical knowledge, that is, the grounds for asserting the truth of mathematical propositions, consists of deductive proof, together with the assumed truth of any premises employed. Apart from the assumed truth of the premises, there is another fundamental way in which mathematical proof depends on truth. The essential underpinning feature of a correct or valid deductive proof is the transmission of truth, that is, truth value is preserved.

Truth in Mathematics

It is often the case in mathematics that the definition of truth is assumed to be clear-cut, unambiguous, and unproblematic. While this is often justifiable as a simplifying assumption, the fact is that it is incorrect and that the meaning of the concept of truth in mathematics has changed significantly over time. I wish to distinguish among three truth-related concepts used in mathematics.

The traditional view of mathematical truth. First of all, there is the traditional view that a mathematical truth is a general statement which not only correctly describes all its instances in the world (as would a true empirical generalization) but is *necessarily* true of its instances. Implicitly underpinning this view is the assumption that mathematical theories have an intended interpretation, often an idealization of some aspect of the world. The key feature of this view is the association of an intended interpretation with a theory. Thus number theory refers to the domain of natural numbers, geometry refers to ideal objects in space, calculus largely refers to functions of the real line, and so on. To be true in this first sense (I will denote it by "$truth_1$") is to be true in the intended interpretation. The mode of expression I have used depends of course upon a modern way of thinking, for it requires prizing open mathematical signs to separate the signifiers (formal mathematical symbols) from the signified (the intended meanings). $Truth_1$ treats mathematical signs as integral; only one interpretation is built in.

$Truth_1$ is analogous to naive realism, a view of truths as statements which accurately describe a state of affairs in some fixed realm of discourse. According to this view, the terms involved in expressing the truth name objects in the intended universe of discourse, and the true statement as a whole describes the relationship that holds between these denotations. In essence, this is the naive correspondence theory of truth.

Such a view of mathematical truth was widespread, dominant even, until the middle and end of the nineteenth century. For example De Morgan commenting on Peacock's new generalized formal algebra described it as made up of "symbols bewitched . . . running about the world in search of meaning" (1835, 311). What he objected to was the severance of algebraic symbols from their generalized arithmetical meanings (Richards 1987). Without such fixed and determinate meanings, mathematical propositions could not express their intended meanings, let alone truths. Similarly, Frege had a sophisticated and philosophically well elaborated view that the theorems of arithmetic are true in its intended interpretation, the domain of natural number. Again, this is the notion of $truth_1$.

Mathematical truth as satisfiability. Secondly, there is the modern view of the truth of a mathematical statement relative to a background mathematical theory: the statement is satisfied by some interpretation or model of the theory. I shall term this second conception "$truth_2$." According to this (and the following) view, mathematical theories are open to multiple interpretations, that is, possible worlds. Truth in this sense consists merely in being true (i.e., satisfied, following Tarski 1936) in one of these possible worlds; that is, in having a model. Thus $truth_2$ is represented by Tarski's explication of truth, which forms the basis of model theory. A proposition is $true_2$ relative to a given mathematical theory if there is some interpretation of the theory which satisfies the proposition, irrespective of the other properties of the interpretation, such as resemblance to some original intended interpretation. (This interpretation must include an assignment of objects and relations of appropriate type to the extralogical symbols, as well as an assignment of values from the universe of discourse to the variable letters of the proposition.)

$Truth_2$ probably originates with Hilbert's work on geometry. Hilbert detached geometrical notions such as 'point, line, and plane' from their original physical (or ideal) interpretations, and argued instead that they could be interpreted as 'table, chair, and beer-mug', provided that what resulted was a model of the axioms of geometry. It has been suggested that Tarski's theory of truth originates in algebra, by analogy with a set of roots satisfying an equation. Likewise, the assignment of values to the components of a proposition satisfies it when it makes it true.

$Truth_2$ is anticipated by Leibniz's notion of 'true in a possible world', which he contrasted with 'true in all possible worlds' (Barcan Marcus 1967).

Logical truth or validity in mathematics. Thirdly, there is the modern view of the logical truth or validity of a mathematical statement relative to a background theory: the statement is satisfied by *all* interpretations or models of the theory. Thus the statement is true in all of these representations of possible worlds. I shall denote this conception of truth by '$truth_3$'. Evidently $truth_3$ more or less corresponds to Leibniz's notion of 'true in all possible worlds'. This is also one of the notions explicated by Tarski's theory of mathematical truth as 'logical validity'.

$Truth_3$ can be established by logical deduction from the background theory if the theory is represented by a first-order axiom set, as Gödel's (1930) completeness theorem establishes. For a given theory, $Truths_3$ (the set of propositions which are true in the sense of $truth_3$) is a subset (usually a proper subset) of $Truths_2$. Incompleteness arises, as Gödel ([1931] 1967) proved, in most mathematical theories as there are $true_1$ sentences (i.e., satisfied in the intended model) which are not $true_3$ (i.e., true in *all* models).

Thus not only does the concept of truth have multiple meanings, but crucial mathematical issues hinge upon this ambiguity. The modern mathematical views of truth ($truth_2$ and $truth_3$) differ in meaning and properties from the traditional mathematical view of $truth_1$ and the everyday naive notion which resembles it. Historically, the transition from $truth_1$ to the modern notions was highly problematic, as Richards (1980, 1989) shows in her studies. Even the correspondence between such mathematically (and philosophically) great thinkers as Frege (1980) and Hilbert shows disagreements and sometimes a lack of understanding that may be attributed to Frege's use of $truth_1$ and Hilbert's use of $truth_2$.