CONCEPTS IN Topology

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Preface

Topology is a major area of mathematics concerned with properties that are preserved undercontinuous deformations of objects, such as deformations that involve stretching, but no tearing or gluing, although the notion of stretching employed in mathematics is not quite the everyday notion: see below and the definition of homeomorphism for details of the mathematical notion. Topology emerged through the development of concepts from geometry and set theory, such as space, dimension, and transformation. Ideas that are now classified as topological were expressed as early as 1736. Toward the end of the 19th century, a distinct discipline developed, which was referred to in Latin as the *geometria situs* ("geometry of place") or *analysis situs* (Greek-Latin for "picking apart of place"). This later acquired the modern name of topology. By the middle of the 20th century, topology had become an important area of study within mathematics. The word *topology* is used both for the mathematical discipline and for a family of sets with certain properties that are used to define a topological space, a basic object of topology. Of particular importance are *homeomorphisms,* which can be defined as continuous functions with a continuousinverse. Topology includes many subfields. The most basic and traditional division within topology is point-set topology, which establishes the foundational aspects of topology and investigates concepts inherent to topological spaces (basic examples include compactness and connectedness); algebraic topology, which generally tries to measure degrees of connectivity using algebraic constructs such as homotopy groups and homology; and geometric topology, which primarily studies manifolds and their embeddings (placements) in other manifolds. Some of the most active areas, such as low dimensional topology and graph theory, do not fit neatly in this division. Knot theory studies mathematical knots.

Topology began with the investigation of certain questions in geometry. Leonhard Euler's 1736 paper on the Seven Bridges of

Königsberg is regarded as one of the first academic treatises in modern topology. The term "Topologie" was introduced in German in 1847 by Johann Benedict Listing in *Vorstudien zur Topologie*, who had used the word for ten years in correspondence before its first appearance in print. "Topology," its English form, was first used in 1883 in Listing's obituary in the journal Nature to distinguish "qualitative geometry from the ordinary geometry in which quantitative relations chiefly are treated". The term topologist in the sense of a specialist in topology was used in 1905 in the magazine *Spectator*. However, none of these uses corresponds exactly to the modern definition of topology. Modern topology depends strongly on the ideas of set theory, developed by Georg Cantor in the later part of the 19th century. Cantor, in addition to establishing the basic ideas of set theory, considered point sets in Euclidean space as part of his study of Fourier series. Henri Poincaré published *Analysis Situs* in 1895, introducing the concepts of homotopy and homology, which are now considered part of algebraic topology. Maurice Fréchet, unifying the work on function spaces of Cantor, Volterra, Arzelà, Hadamard, Ascoli, and others, introduced the metric space in 1906. A metric space is now considered a special case of a general topological space. In 1914, Felix Hausdorff coined the term "topological space" and gave the definition for what is now called a Hausdorff space. In current usage, a topological space is a slight generalization of Hausdorff spaces, given in 1922 by Kazimierz Kuratowski.

The texts are arranged in a lucid form and written in colloquial English. All the essential aspects of this subject have been included. Hopefully, the present study will prove very useful for students and teachers.

—Editor

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1

Introduction

Topology, as a branch of mathematics, can be formally defined as "the study of qualitative properties of certain objects (called topological spaces) that are invariant under a certain kind of transformation (called a continuous map), especially those properties that are invariant under a certain kind of equivalence (called homeomorphism)." To put it more simply, topology is the study of continuity and connectivity.

The term *topology* is also used to refer to a structure imposed upon a set X, a structure that essentially 'characterizes' the set X as a topological space by taking proper care of properties such as convergence, connectedness and continuity, upon transformation.

Topological spaces show up naturally in almost every branch of mathematics. This has made topology one of the great unifying ideas of mathematics. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. For example, the square and the circle have many properties in common: they are both one dimensional objects (from a topological point of view) and both separate the plane into two parts, the part inside and the part outside.

One of the first papers in topology was the demonstration, by Leonhard Euler, that it was impossible to find a route through the town of Königsberg (now Kaliningrad) that would cross each of its seven bridges exactly once. This result did not depend on the lengths of the bridges, nor on their distance from one another, but only on connectivity properties: which bridges are connected to which islands or riverbanks. This problem, the *Seven Bridges of*

Königsberg, is now a famous problem in introductory mathematics, and led to the branch of mathematics known as graph theory.

Similarly, the hairy ball theorem of algebraic topology says that "one cannot comb the hair flat on a hairy ball without creating a cowlick." This fact is immediately convincing to most people, even though they might not recognize the more formal statement of the theorem, that there is no nonvanishing continuous tangent vector field on the sphere. As with the *Bridges of Königsberg*, the result does not depend on the exact shape of the sphere; it applies to pear shapes and in fact any kind of smooth blob, as long as it has no holes.

To deal with these problems that do not rely on the exact shape of the objects, one must be clear about just what properties these problems *do* rely on. From this need arises the notion of homeomorphism. The impossibility of crossing each bridge just once applies to any arrangement of bridges homeomorphic to those in Königsberg, and the hairy ball theorem applies to any space homeomorphic to a sphere.

Intuitively two spaces are homeomorphic if one can be deformed into the other without cutting or gluing. A traditional joke is that a topologist can't distinguish a coffee mug from a doughnut, since a sufficiently pliable doughnut could be reshaped to the form of a coffee cup by creating a dimple and progressively enlarging it, while shrinking the hole into a handle. A precise definition of homeomorphic, involving a continuous function with a continuous inverse, is necessarily more technical.

Homeomorphism can be considered the most basic *topological equivalence*. Another is homotopy equivalence. This is harder to describe without getting technical, but the essential notion is that two objects are homotopy equivalent if they both result from "squishing" some larger object.

An introductory exercise is to classify the uppercase letters of the English alphabet according to homeomorphism and homotopy equivalence. The result depends partially on the font used. The figures use a sans-serif font named Myriad. Notice that homotopy equivalence is a rougher relationship than homeomorphism; a homotopy equivalence class can contain several of the homeomorphism classes. The simple case of homotopy equivalence described above can be used here to show two letters are homotopy

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equivalent. For example, O fits inside P and the tail of the P can be squished to the "hole" part. Thus, the homeomorphism classes are: one hole two tails, two holes no tail, no holes, one hole no tail, no holes three tails, a bar with four tails (the "bar" on the *K* is almost too short to see), one hole one tail, and no holes four tails.

The homotopy classes are larger, because the tails can be squished down to a point. The homotopy classes are: one hole, two holes, and no holes. To be sure we have classified the letters correctly, we not only need to show that two letters in the same class are equivalent, but that two letters in different classes are not equivalent. In the case of homeomorphism, this can be done by suitably selecting points and showing their removal disconnects the letters differently. For example, X and Y are not homeomorphic because removing the centre point of the X leaves four pieces; whatever point in Y corresponds to this point, its removal can leave at most three pieces. The case of homotopy equivalence is harder and requires a more elaborate argument showing an algebraic invariant, such as the fundamental group, is different on the supposedly differing classes.

Letter topology has some practical relevance in stencil typography. The font Braggadocio, for instance, has stencils that are made of one connected piece of material.

Mathematical Definition

Let X be a set and let τ be a family of subsets of X. Then τ is called a *topology on* X if:

1. Both the empty set and X are elements of τ
2. Any union of elements of τ is an element of τ
3. Any intersection of finitely many elements of t is an element of τ.

If τ is a topology on X, then the pair (X, τ) is called a *topological space*. The notation X_τ may be used to denote a set X endowed with the particular topology τ.

The members of τ are called *open sets* in X. A subset of X is said to be closed if its complement is in τ (i.e., its complement is open). A subset of X may be open, closed, both (clopen set), or neither. The empty set and X itself are always clopen.

A function or map from one topological space to another is called *continuous* if the inverse image of any open set is open. If the function maps the real numbers to the real numbers (both spaces with the Standard Topology), then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto, and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. Another way of saying this is that the function has a natural extension to the topology. If two spaces are homeomorphic, they have identical topological properties, and are considered topologically the same. The cube and the sphere are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut.

Topology Topics

Some Theorems in General Topology

- Every closed interval in R of finite length is compact. More is true: In R^n, a set is compact if and only if it is closed and bounded.
- Every continuous image of a compact space is compact.
- Tychonoff's theorem: the (arbitrary) product of compact spaces is compact.
- A compact subspace of a Hausdorff space is closed.
- Every continuous bijection from a compact space to a Hausdorff space is necessarily a homeomorphism.
- Every sequence of points in a compact metric space has a convergent subsequence.
- Every interval in R is connected.
- Every compact finite-dimensional manifold can be embedded in some Euclidean space R^n.
- The continuous image of a connected space is connected.
- Every metric space is paracompact and Hausdorff, and thus normal.
- The metrization theorems provide necessary and sufficient conditions for a topology to come from a metric.
- The Tietze extension theorem: In a normal space, every continuous real-valued function defined on a closed

subspace can be extended to a continuous map defined on the whole space.

- Any open subspace of a Baire space is itself a Baire space.
- The Baire category theorem: If X is a complete metric space or a locally compact Hausdorff space, then the interior of every union of countably many nowhere dense sets is empty.
- On a paracompact Hausdorff space every open cover admits a partition of unity subordinate to the cover.
- Every path-connected, locally path-connected and semi-locally simply connected space has a universal cover.

General topology also has some surprising connections to other areas of mathematics. For example:

- In number theory, Fürstenberg's proof of the infinitude of primes.

Some useful notions from algebraic topology

- *Homology and cohomology:* Betti numbers, Euler characteristic, degree of a continuous mapping.
- *Operations:* cup product, Massey product
- *Intuitively attractive applications:* Brouwer fixed-point theorem, Hairy ball theorem, Borsuk–Ulam theorem, Ham sandwich theorem.
- Homotopy groups (including the fundamental group).
- Chern classes, Stiefel–Whitney classes, Pontryagin classes.

Generalizations

Occasionally, one needs to use the tools of topology but a "set of points" is not available. In pointless topology one considers instead the lattice of open sets as the basic notion of the theory, while Grothendieck topologies are certain structures defined on arbitrary categories that allow the definition of sheaves on those categories, and with that the definition of quite general cohomology theories.

Simplex

In geometry, a simplex (plural *simplexes* or *simplices*) is a generalization of the notion of a triangle or tetrahedron to arbitrary dimension. Specifically, an n-simplex is an n-dimensional polytope which is the convex hull of its $n + 1$ vertices. For example, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, and a 4-simplex is a pentachoron.

A single point may be considered a 0-simplex, and a line segment may be considered a 1-simplex. A simplex may be defined as the smallest convex set containing the given vertices.

Figure: A regular 3-simplex or tetrahedron

A regular simplex is a simplex that is also a regular polytope. A regular n-simplex may be constructed from a regular $(n - 1)$-simplex by connecting a new vertex to all original vertices by the common edge length. In topology and combinatorics, it is common to "glue together" simplices to form a simplicial complex.

The associated combinatorial structure is called an abstract simplicial complex, in which context the word "simplex" simply means any finite set of vertices.

The Standard Simplex

The standard n-simplex (or unit n-simplex) is the subset of R^{n+1} given by

$$\Delta^n = \left\{(t_0, \cdots, t_n) \in \mathbb{R}^{n+1} \mid \Sigma_{i=0}^n t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\right\}$$

The simplex Δ^n lies in the affine hyperplane obtained by removing the restriction $t_i \geq 0$ in the above definition. The standard simplex is clearly regular.

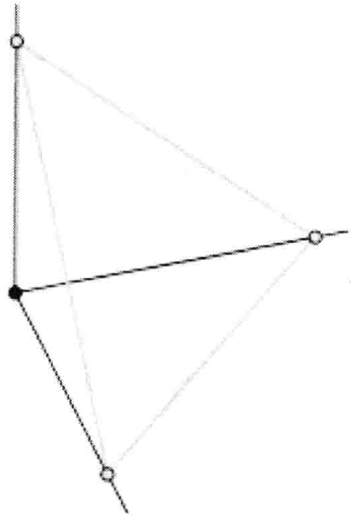

Figure: *The Standard 2-simplex in* R^3

The n+1 vertices of the standard n-simplex are the points e_i – R^{n+1}, where

$e_0 = (1, 0, 0, ..., 0),$

$e_1 = (0, 1, 0, ..., 0),$

$\vdots$

$e_n = (0, 0, 0, ..., 1).$

There is a canonical map from the standard n-simplex to an arbitrary n-simplex with vertices (v_0, ..., v_n) given by

$$(t_0, \cdots, t_n) \mapsto \Sigma_{i=0}^{n} t_i v_i$$

The coefficients t_i are called the barycentric coordinates of a point in the n-simplex. Such a general simplex is often called an affine n-simplex, to emphasize that the canonical map is an affine transformation. It is also sometimes called an oriented affine n-simplex to emphasize that the canonical map may be orientation preserving or reversing.

More generally, there is a canonical map from the standard $(n-1)$-simplex (with n vertices) onto any polytope with n vertices, given by the same equation (modifying indexing):

$$(t_1, \cdots, t_n) \mapsto \Sigma_{i=1}^{n} t_i v_i$$

These are known as generalized barycentric coordinates, and express every polytope as the *image* of a simplex: $\Delta^{n-1} \twoheadrightarrow P$.

Increasing Coordinates

An alternative coordinate system is given by taking the indefinite sum:

$$
\begin{aligned}
s_0 &= 0 \\
s_1 &= s_0 + t_0 = t_0 \\
s_2 &= s_1 + t_1 = t_0 + t_1 \\
s_3 &= s_2 + t_2 = t_0 + t_1 + t_2 \\
&\ldots \\
s_n &= s_{n-1} + t_{n-1} = t_0 + t_1 + \ldots + t_{n-1} \\
s_{n+1} &= s_n + t_n = t_0 + t_1 + \ldots + t_n = 1
\end{aligned}
$$

This yields the alternative presentation by *order,* namely as nondecreasing n-tuples between 0 and 1:

$$\Delta_*^n = \left\{(s_1, \cdots, s_n) \in \mathbb{R}^n \mid 0 = s_0 \leq s_1 \leq s_2 \leq \ldots \leq s_n \leq s_{n+1} = 1\right\}.$$

Geometrically, this is an n-dimensional subset of $\mathbb{R}^n$ (maximal dimension, codimension 0) rather than of $\mathbb{R}^{n+1}$ (codimension 1). The hyperfaces, which on the standard simplex correspond to one coordinate vanishing, $t_i = 0$, here correspond to successive coordinates being equal, $s_i = s_{i+1}$, while the interior corresponds to the inequalities becoming *strict* (increasing sequences).

A key distinction between these presentations is the behaviour under permuting coordinates – the standard simplex is stabilized by permuting coordinates, while permuting elements of the "ordered simplex" do not leave it invariant, as permuting an ordered sequence generally makes it unordered.

Indeed, the ordered simplex is a (closed) fundamental domain for the action of the symmetric group on the n-cube, meaning that the orbit of the ordered simplex under the $n!$ elements of the symmetric group divides the n-cube into $n!$ mostly disjoint simplices (disjoint except for boundaries), showing that this simplex has volume $1/n!$ Alternatively, the volume can be computed by an iterated integral, whose successive integrands are $1, x, x^2/2, x^3/3!, \ldots, x^n/n!$

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