AN INTRODUCTION TO FLUID DYNAMICS

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Dedicated to my parents

Shri Sunder Dass Rathy and
Smt. Laxmi Devi Rathy

FOREWORD

One of the oldest branches of applied mathematics is Fluid Dynamics. It is also the branch in which some of the most significant advances have been made during the last fifty years. These advances have been motivated by exciting developments in science and technology and have been facilitated by growth of computer capabilities and developments of sophisticated mathematical techniques. Thus non-Newtonian fluid flows have been discussed in this period because these new types of fluids are being used by chemical industries. Magnetohydrodynamics has been developed because of the astrophysical phenomena arising out of the discovery of existence of large magnetic fields in some stars and the possibilities of magnetohydrodynamic generators as a new source of energy. Flows through porous media have been investigated because of their importance in industry and in petroleum technology. The actual development of supersonic aircraft has motivated a great deal of research in supersonic and hypersonic flows.

There has been tremendous research activity in Fluid Dynamics and there are enough challenging problems in the field to keep mathematicians, scientists and engineers busy for another fifty years and in this period new and more challenging problems are bound to arise.

Indian applied mathematicians have made very significant contributions to these researches in Fluid Dynamics. About four hundred candidates have obtained their Ph.D. degrees in this field and on a rough estimate about ten thousand research papers have been published in reputed journals by Indian research workers during the last fifty years. All this has happened in spite of the fact that the only exposure our students had to fluid dynamics was through books by Ramsay, Lamb, Milne-Thompson which treated incompressible inviscid flows in details, but did not give students even on introduction to some of the new topics in Fluid Dynamics. This was due to the reason that these books

were written before these new disciplines began to develop.

I was therefore keen that a textbook should be written which should give an introduction to topics like boundary layer theory, compressible fluid flows, non-Newtonian fluid flows, stability of fluid flows, turbulence theory and magnetohydrodynamics. I suggested this idea to a few of my over one dozen students who had taken their Ph.D. degrees in some of these topics under my guidance. I am glad that Dr. Rathy accepted the challenge and spent a great deal of time and effort in producing the present book. He has taken great pains to make the presentation lucid and to give a large number of exercises which are essential for a textbook. I am also glad that one of our leading publishing firms has undertaken the publication of this book and the National Book Trust, India has agreed to subsidise this publication so that this useful book will now be available to our students at a reasonable price.

I hope this book will be used by all the mathematics departments as well as by aeronautical, civil, mechanical and chemical engineering departments. I hope also that Dr. Rathy will be encouraged to write a second volume discussing present topics at a more advanced level as well as covering topics like flow through porous media, heat transfer, hypersonic flows, biofluid-dynamics etc. I hope students who study this book will find the subject exciting and challenging and will be encouraged to go deeper into the subject.

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PREFACE

Fluid dynamics is an important and rich branch of science. The development of fluid dynamics started in quite early age of civilization. But before this century the discussion in fluid dynamics was mainly confined to the study of inviscid incompressible flows. In fact, the liquids and gases were studied separately and had almost no link between them. The subject concerning the liquids was usually called hydrodynamics. This name in fact appeared plausible as the main subject of study of liquid flows were the water flows. Engineers were interested in discussing the thrust of water in dams and tanks, flows of water in canals, pipes and turbines, etc. From mathematical point of view also this subject was important as the study of flows gave birth to various mathematical techniques and had many challenging problems. In hydrodynamics the main topics to be discussed were the potential flows, gravity waves, vortex motions, sources and sinks. Though the concept of viscosity is as old as time of Newton, who gave Newtonian hypothesis, but this concept could not be incorporated in the study till 1852 when Stokes gave the famous Navier-Stokes equations. Moreover, discussion of viscous fluids could not gain momentum till the early years of this century.

Towards the beginning of the present century a big boost came to this subject with the development of boundary layer theory, turbulence, and subsonic and supersonic flows. Many new branches came in perview of this subject. The distinction between gases and liquids appeared to be an artificial as both the liquids and the gases obey the same basic laws. Thus the study of both the subjects was merged and now the subject is usually called as fluid dynamics. This name gives feeling of a wider scope of discussing the behaviour of all types of fluids. In fact this should still be generalised to subject called continuum mechanics which embrasses both the behaviour of solids and the

fluids. Recent years have witnessed a completely new direction to the subject which is fast developing one and which has put the treatment of this subject on a rational basis. Moreover, many new types of fluids have either become important or have been discovered recently especially with the development of polymer industry. In last few decades the subject has increased at an exponential rate and researches in fluid dynamics are becoming more and more important both from mathematical point of view and also from application point of view.

About a quarter century back there were only few books on fluid dynamics, and very few from mathematical aspect. Whatever books on the mathematical aspect were available were dealing with classical topics, e.g., the potential flows, sources and sinks, the vortex motion and gravity waves. The books dealing with engineering aspect were mainly confined to empirical formulation and so they lacked in giving a connected account of the subject. Therefore there was a strong need for a book which could give a connected and simple account of the subject. In recent years many new books have come out on fluid dynamics. These books are either confined to few special topics in fluid dynamics and are at research level and so are not suitable for a classroom teaching or they do not give a connected view of the subject.

In this book I have tried to give a connected and simple account of the subject. Starting from the basic physical properties of fluids and the relevant thermodynamics, I have developed the basic equation for any fluid flow almost in a self-contained manner. After this I have given an account of classical topics to stress their richness even in present context. This is followed by some simple exact solutions in various fields. Finally, I have discussed some special topics to indicate the wide range of branches of fluid dynamics. In this book I have throughout stressed on the basic concepts rather than the difficult problems. On the other hand the difficult problems have been kept to the minimum so that the reader can adopt this book easily. Another aim of this book is to give a brief account of fluid dynamics as a whole rather than stressing only on few topics. The problems have been designed in such a way that the students can themselves pursue the subject to more or less at the research level.

I have also tried to keep this book almost self-contained. In fact, the student is only expected to know calculus, simple ordinary and partial differential equations and some knowledge of complex variable.

In chapter I, a brief account of basic physical properties of fluid is given and in chapter II a brief account of thermodynamics relevant to fluid flows is given. In chapters III and IV, I have discussed the kinematics of fluids and the stresses. In chapter V the constitutive equations—the relations connecting stresses and strain and strain-rates, have been obtained. In this both the Newtonian and non-Newtonian fluids have been included. In chapter VI, the basic equations governing any fluid flow have been developed from the basic conservation laws. In chapter VII, the inviscid flows, including the potential flows, vortex flows and gravity waves have been discussed. This chapter mainly contains the material usually covered in classical books. I have included only those topics which clarify some basic concepts rather than attempting the difficult problems. So I have omitted the motion of elliptic cylinders, flows through different geometries, flows in multi-connected regions, etc. Even then this chapter happens to be the largest in the book. In chapter VIII, I have discussed some exact solutions of Newtonian fluids, which are the most important fluids. In chapter IX, the flows of compressible fluids and in chapter X the flows of non-Newtonian fluids have been discussed. In chapter XI, the dimensional analysis has been given and the non-dimensional numbers have been discussed. Along with this the importance of these numbers in fluid flows have been stressed. After this the small Reynolds number flows —the creeping flows, flow through porous media and lubrication theory have been discussed. In chapters XII, XIII and XIV, I have discussed some special topics, namely boundary layer theory, stability theory and turbulence. Then I felt that book would remain incomplete without a brief account of hydromagnetics—the branch which has recently become very important. So in chapter XV, I have discussed the hydromagnetic flows.

This book can be used in one year course or in two one-semester courses of three to four hours teaching per week. In two one-semester courses one can choose chapters I to IX in the first semester and remaining six chapters in the second

semester. In fact the later chapters (i.e. X to XV) are more or less independent chapters and one can take one or more of these chapters depending upon one's interest and the time available.

I am extremely grateful to my teacher, Prof. J.N. Kapur, Senior Professor of Mathematics, I.I.T., Kanpur, formerly Vice-Chancellor, Meerut University, Meerut, for his constant encouragement without which this book would have not been there. I am also grateful to my wife, Suman Rathy, for creating a perfect homely atmosphere. In the last, but not the least, I am grateful to M/s Oxford & IBH Publishing Co. for their excellent cooperation and keenness in bringing out this book.

R.K. RATHY

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