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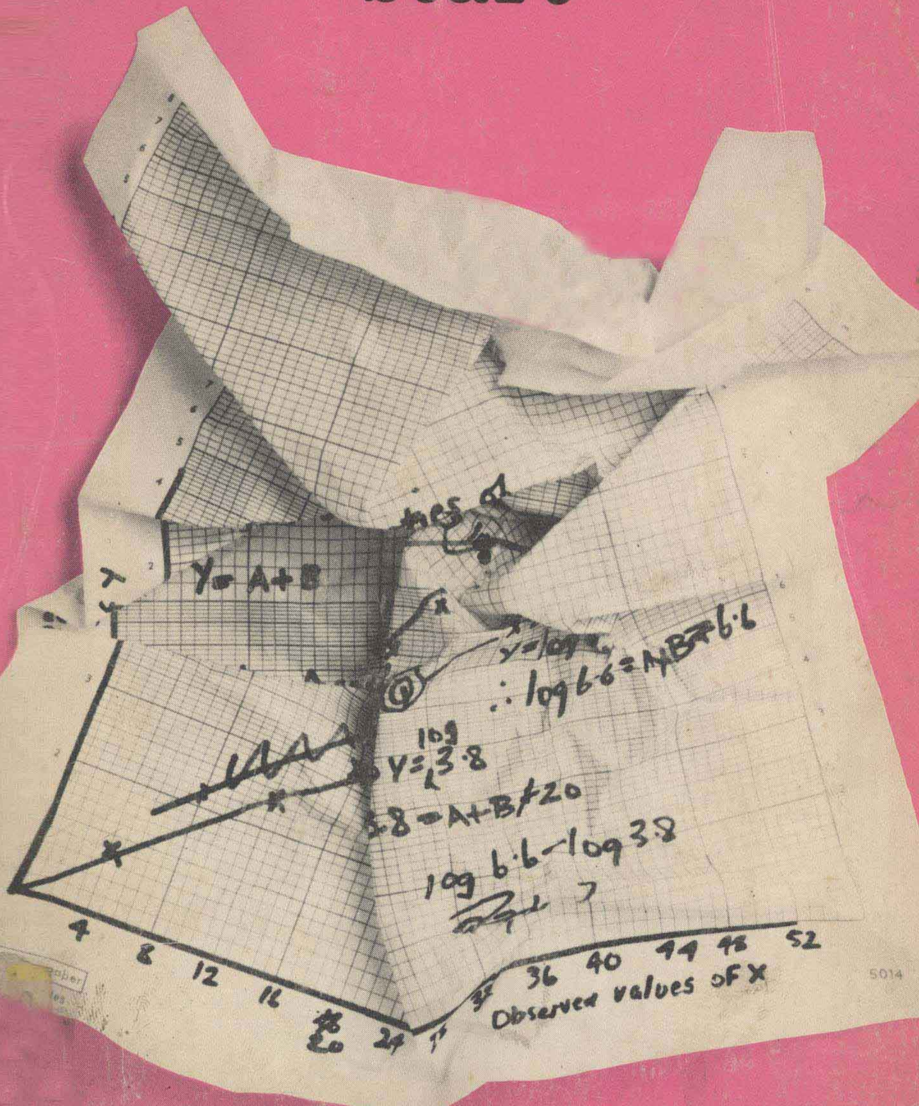
MATHEMATICS

a second start



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MATHEMATICS
a second start

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Longman
London and New York

Longman Group Limited London

Associated companies, branches and representatives
throughout the world

Published in the United States of America
by Longman Inc., New York

First published 1979

Library of Congress Cataloging in Publication Data

Page, S G
Mathematics : a second start.
1. Mathematics - 1961 - I. Title.
QA37.2.P33 510 77-21205
ISBN 0-582-41755-4

Longman

Printed in Hong Kong by
Commonwealth Printing Press Ltd

INTRODUCTION

This book is intended for the student 'who never could do maths', the student who for one reason or another missed his or her way at school, either by absence from some of the O-level course, or by having too many changes of masters or schools. Such a student has lost confidence in his own ability to tackle the subject. It is intended for the student whose knowledge extends only to O-level mathematics, probably obtained several years ago, but who would now like to be able to communicate with mathematicians, i.e. to know what is meant when one talks about 'an integral' or 'a differential equation'. It is based on a course of tutorials and discussions with students of this type, to whom I am indebted for their patient endeavours and encouragement.

So let us begin at the beginning. The student who does not require the elementary work, nevertheless, is advised to see that the examples at the end of each chapter are worked through conscientiously, as each step depends on the previous steps being fully understood. One must be completely competent in dealing with mathematical 'shorthand', i.e. the method of expressing an idea in as neat a way as possible, so that it can be handled easily and quickly.

CONTENTS

Chapter 1
Learning the language

You will know that the area of a rectangle is found by multiplying the length by the breadth, or more shortly

$A = L \times B$
or $A = L.B$
or $A = LB$

This simple shorthand is expressing the fact that 'if you multiply the length of a rectangle by the breadth, both being expressed in the same unit, then the result gives the area of the rectangle in square units'.

This is algebra, and the first fact emerges:

When two symbols are written side by side with no sign in between, **the sign understood is multiplication.** Multiplication is the **only sign** which can be omitted between two symbols and then **only between letters** or a letter and a number but not between two numbers.

$a \times b = ab$
But 2×3 is obviously not 23

We also know the shorthand way of writing
$2 \times 2 \times 2 \times 2 \times 2$ is 2^5
Similarly
$a \times a \times a \times a \times a = a^5$
Now we can distinguish
$2a$ from a^2
$2a$ means $2 \times a$
a^2 means $a \times a$ } THIS IS VERY IMPORTANT

i.e. if $a = 5$, $2a = 10$ but $a^2 = 25$

Similarly $3b$ means $3 \times b$
but b^3 means $b \times b \times b$

Exercise 1.

1. Write down the shorthand form for:

(i) $a \times a \times a \times a$ (iii) $2 \times a \times b \times c$
(ii) $4 \times a$ (iv) $3 \times a \times b \times b \times c \times c \times c$

2. If $a = 2$ and $b = 3$, find the value of

(i) a^2, $3a$ (iv) $3a^2 b$
(ii) b^2, $2b$ (v) $4ab$
(iii) ab (vi) $2a + 3b$

INDICES

Knowing the meaning of a^2, a^3, a^4, etc., we can now multiply powers of the same letter (or number) together.

e.g. $a^4 \times a^3$ means $(a \times a \times a \times a)$ multiplied by $(a \times a \times a)$
i.e. $(a \times a \times a \times a) \times (a \times a \times a) = a \times a \times a \times a \times a \times a \times a$

Therefore $a^4 \times a^3$ is a^7.

So we **ADD THE INDICES when multiplying powers of the same letter or number.**

THIS IS THE FIRST RULE OF INDICES

This helps us to do quickly what we know by common sense is the meaning of a^4 and a^3 and we can imagine a row of seven a's all multiplied together. There is no mystery about this rule.

Similarly $a^6 \div a^2$ means $\dfrac{a \times a \times a \times a \times a \times a}{a \times a}$

Whatever number the a's represent, they all stand for the same number in the same question, so that two of the a's cancel out giving:

$$\frac{\overset{1}{\cancel{a}} \times \overset{1}{\cancel{a}} \times a \times a \times a \times a}{\underset{1}{\cancel{a}} \times \underset{1}{\cancel{a}}} = \frac{1\,a^4}{1} = a^4$$

This illustrates the rule of **division of powers** of the same letter or number, i.e. we **SUBTRACT THE INDEX of the bottom line from that of the top line.**

Exercise 2. Simplify:

1. $a^6 \times a^2$
2. $a^2 \times a^2 \times a$ (note a is the same as a^1)
3. $a^5 \times a \times a^3$
4. $a^7 \div a^3$
5. $a^9 \div a^3$ (be careful!)
6. $a^{10} \div a^5$
7. $ab^4 \times a^2b^2$ (multiply the a's and then the b's)
8. $a^2b^3 \times a^2b^2 \div ab$

(Write No.8 out in full, cancel and then simplify).

Some Common Errors

We must distinguish between the meanings of $2a^2$ and $(2a)^2$. The former is 'two a-squared' and the latter is 'two-a all squared', and the difference is fundamental.

For example:

if 'a' represents the number 4
$2a^2$ (which is $2 \times a \times a$) means $2 \times 4 \times 4 = 32$
but $(2a)^2$ (meaning $2a \times 2a$) means $8 \times 8 = 64$, i.e. $(2a)^2 = 4a^2$

so that the 'square' **only affects the letter or number on which it stands, but when a bracket is squared, everything inside** is squared.

Exercise 3. Simplify:

1. $(2a)^3$ 2. $(3ab)^2$ 3. $(ab)^3 \times (a^2b)^2$

ADDITION AND SUBTRACTION

Having coped with multiplication and division, let us add and subtract. We know that three 2's are 6 and four 2's are 8 and 6 + 8 = 14 = seven 2's, so that three 2's + four 2's make seven 2's.

This is a particular case of $3a + 4a = 7a$

3 x (a particular thing) + 4 x (the same kind of thing)
= 7 x (the same kind of thing)

Don't make the mistake of saying $3a + 4a = 7a^2$ which is a common error.

3 apples + 4 apples = 7 apples (not apples squared)

In the same way:

10 apples − 4 apples = 6 apples

i.e. $10a - 4a = 6a$

Now we can add and subtract in algebra. It is as simple as this.

$12a + 3a + a = 16a$ (a is the same as 1a)

and $5y + 3y - 2y = 6y$

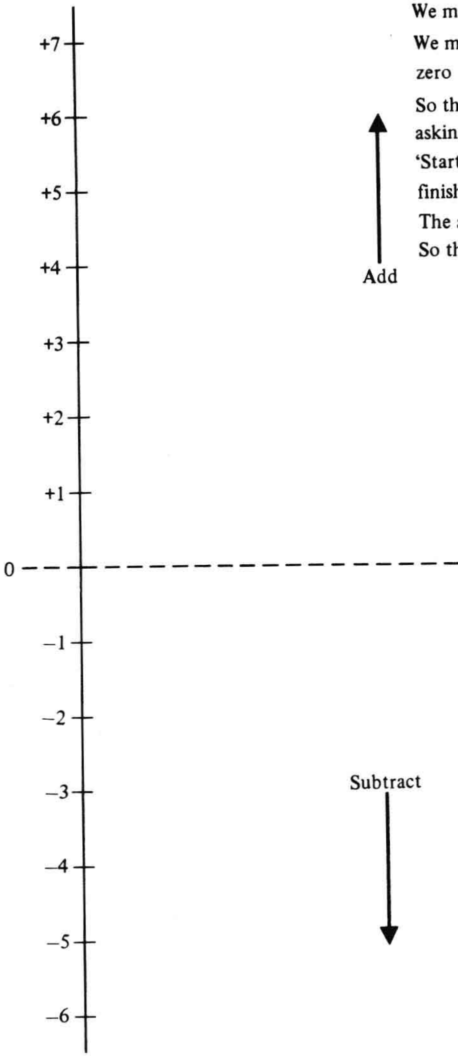
Exercise 4. Write down the results of the following additions and subtractions:

1. $2x + 7x$
2. $5x - 2x$
3. $14x - 11x$
4. $9x + 2x - x$
5. $12a + 5a - 7a$
6. $13x^2y + 2x^2y - 5x^2y$

NEGATIVE NUMBERS

Now consider $5a - 7a$.
This is not the same as $7a - 5a$ (though $5a + 7a$ is the same as $7a + 5a$, both equalling $12a$).

We can consider $5a - 7a$ in several different ways. One popular way is the 'thermometer scale' method (Fig.1.1).



We measure plus numbers upwards from zero
We measure minus numbers downwards from zero
So that $5a - 7a$ is the shorthand way of asking
'Start at 5 and go down 7 and where do you finish?'
The answer is -2
So that $5a - 7a = -2a$

Figure 1.1

or

One can think of *plus* terms pulling one way and *minus* terms pulling the opposite way (Fig.1.2).

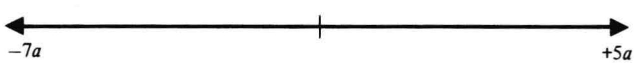


Figure 1.2

So that $+ 5a$ and $- 7a$ are pulling in opposite directions giving the result that the negative term 'wins'. And by how much? $- 2a$. So that $+ 5a - 7a = - 2a$.

Note 1: The only sign which can be omitted in front of a term is a plus sign.

Note 2: The only sign which can be omitted between two letters is the multiplication sign.

Note 3: Only terms which are exactly the same type of term can be added or subtracted: e.g. one cannot add or subtract terms like $2a^2b$ and $3ab^2$ although they appear similar.

a^2b means $a \times a \times b$
and
ab^2 means $a \times b \times b$

These are quite different as will be apparent if values are put in for a and b, but a^2b is equal to ba^2 as the order of multiplication does not affect the result:

e.g. $3^2 \times 2$ is the same as 2×3^2

Example 1. Simplify: $- 7a - 3a$

These terms, $- 7a$ and $- 3a$, are both pulling in the same direction, the negative direction.
Therefore, they combine to make $- 10a$.
We say, $- 7a - 3a = - \mathbf{10a}$.

Example 2. Simplify: $5x - 3x - x + 3x$

We can perform this operation in any order, provided we remember that **the sign IN FRONT OF a term belongs to the term.**

Hence $5x - 3x - x + 3x$

$= 2x - x + 3x$

$= x + 3x$

$= \mathbf{4x}$

or $5x - 3x - x + 3x$

$= +8x - 4x$ Combining the plus terms and the minus terms first.

$= \mathbf{4x}$

Exercise 5. Collect like terms in the following:

(Example: $-2a - 3b + 4c + a - 6b + 5c = -a - 9b + 9c$)

1. $3a - 4b + 2c - 6b + 2a - 5c$
2. $5x + 2y - 9z - 7x - 5y + 12z$
3. $11x^2y + 2x^2y - 6x^2y - 12x^2y$
4. $-6pq + 11pq - 3pq$
5. $6a - 7b + 2a - 7b + 14b - 8a$
6. Add up the following columns:

(a)
$$\begin{array}{r} 2x - 3y + 7z \\ -2x - 2y - 3z \\ 6x + y - z \\ \hline \\ \hline \end{array}$$

(b)
$$\begin{array}{r} 3x^2 - 2x + 4 \\ -3x^2 - 5x - 1 \\ -6x^2 + 3x - 5 \\ \hline \\ \hline \end{array}$$

Worked Examples on Chapter 1. Simplify:

(i) $1\frac{1}{4}y \times 8$

$$= \frac{5}{\not{4}_1} \times \frac{y}{1} \times \frac{\not{8}^2}{1}$$

$= 10y$ (after cancelling by 4)

(ii) $\frac{4x^2}{9} \div \frac{2x^2}{3}$

$$= \frac{{}^2\not{4}x^2}{\not{9}_3} \times \frac{\not{3}^1}{\not{2}x^2_1}$$ (change ÷ to x and invert the second fraction)

$= \frac{2x^2}{3x^2}$ (after cancelling by 2 and 3)

$= \frac{2}{3}$ (after cancelling by x^2)

(iii) $\frac{p}{q} \times \frac{q}{r} \times \frac{r}{s} \div \frac{p}{s}$

$$= \frac{{}^1\not{p}}{{}_1\not{q}} \times \frac{{}^1\not{q}}{{}_1\not{r}} \times \frac{{}^1\not{r}}{{}_1\not{s}} \times \frac{{}^1\not{s}}{{}_1\not{p}}$$

$= \frac{1}{1}$ (everything cancels)

$= 1$ *

(* Don't make the error and say that 'everything cancels out ∴ the answer is nought').

Exercise 6. Simplify:

1. $3\frac{1}{3}ab \times 9$

2. $\frac{2a}{b} \times \frac{c}{4a} \times \frac{6b}{c}$

3. $\frac{6a^2}{5} \div \frac{3ab}{10}$

4. $\frac{a}{b} \times \frac{2a}{b} \div \frac{3a}{2c}$

5. $\frac{5a}{b} \times \frac{3b}{4c} \div (\frac{2a}{3} \times \frac{b}{10c})$

6. $\frac{6}{5p} \times \frac{2q^2}{3q} \div \frac{q}{10r^2}$

Chapter 2

Useful tools

EXPRESSIONS INVOLVING BRACKETS

You are probably familiar with the guide to simplifying complicated expressions in arithmetic - B,O,D,M,A,S, meaning that the order of simplification is:

1. Brackets and 'Of'
   then
2. Division and Multiplication
   and lastly
3. Addition and Subtraction

Our only concern here is that **brackets have priority over other operations,** meaning that the contents of **a bracket must be simplified if possible before any other operation is performed.**

In algebra it is not always possible to simplify the contents of a bracket. Therefore we have to find other ways of removing the bracket without changing the value of the expression.

Example. How do we simplify $2(a + b) + 3(2a + 4b)$?
Let us consider $2(3 + 7)$
We know that this means $2 \times (3 + 7)$
(The only sign that can be omitted between two things is the multiplication sign)
So that $2 \times (3 + 7) = 2 \times 10 = 20$
Can we get the same answer any other way?

Can we split up the expression $2 \times (3 + 7)$?
Let us try $(2 \times 3) + 7$. This gives $6 + 7 = 12$, which is *incorrect.*

BUT (2 x 3) + (2 x 7) gives 6 + 14 = 20, i.e. the correct answer.
This suggests that **everything INSIDE the bracket must be multiplied by the quantity OUTSIDE the bracket.**
Let us consider another case: 5(60 – 3 + 5)
By conventional arithmetic this is 5 x 62 = 310
Let us try the other method, multiplying each number inside by 5 separately:
(5 x 60) – (5 x 3) + (5 x 5)
= 300 – 15 + 25
= **310**
It gives the correct answer although it is a longer method.

BRACKETS IN ALGEBRA

Following the same rule, for our first example:
2(a + b) + 3(2a + 4b)
gives 2a + 2b + 6a + 12b, multiplying the first bracket by 2 and the second by 3
= **8a + 14b**

Similarly 5(6p + 4q) + 2(p – 3q)
= 30p + 20q + 2p – 6q
= **32p + 14q**
And 4(a + b + c) + (2a – 3b – 6c)
= 4(a + b + c) + 1(2a – 3b – 6c)
= 4a + 4b + 4c + 2a – 3b – 6c
= **6a + b – 2c**

Note: If there is NO NUMBER OR TERM OUTSIDE a bracket, the number is understood to be 1.

NEGATIVE QUANTITIES OUTSIDE A BRACKET

Let us consider the value of
3 x (4 + 1) – 2 x (2 + 4)
= 3 x 5 – 2 x 6
= 15 – 12
= **3**

By the long method, would the following give the correct answer?

$$(3 \times 4) + (3 \times 1) - (2 \times 2) + (2 \times 4)$$

Simplifying:

$$12 + 3 - 4 + 8$$
$$= 23 - 4$$
$$= 19 \text{ } \textit{wrong}$$

Note the change of sign here

Let us try again:

$$(3 \times 4) + (3 \times 1) - (2 \times 2) - (2 \times 4)$$
$$= 12 + 3 - 4 - 8$$
$$= 3 \text{ } \textit{correct}$$

This seems to indicate that a NEGATIVE SIGN OUTSIDE THE BRACKET REVERSES the signs INSIDE when the bracket is removed.

Let us try another example:

$$3 \times (4 - 1) - 2 \times (4 - 2)$$
$$= (3 \times 3) - (2 \times 2) \qquad = 5$$

Using the long way and changing the sign on removing the second bracket:

$$3 \times (4 - 1) - 2(4 - 2)$$
$$= 3 \times 4 - 3 \times 1 - 2 \times 4 + 2 \times 2$$
$$= 12 - 3 - 8 + 4 \qquad = 5 \text{ correct.}$$

Note the change of sign here.

This gives the following:

RULES FOR THE REMOVAL OF SIMPLE BRACKETS

1. **Every term INSIDE** the bracket must be multiplied by the number or quantity **OUTSIDE** the bracket.
2. If the sign **in FRONT of** the bracket is **POSITIVE** the signs **INSIDE** the bracket are **unchanged.** (Note: No sign in front of the bracket implies that it is positive).
3. If the sign **in FRONT of** the bracket is **NEGATIVE all the signs INSIDE are changed** from positive to negative or negative to positive.

Exercise 7. Remove brackets from the following and collect like terms:

1. $3(a + b) + 4(5a + b)$
2. $3(a - b) + 4(2a + b)$
3. $6(a + 2b - c) + (a - b + c)$
4. $2(3a - 4b) - (a + b) + 2(a + b)$
5. $3(a - 2b + 3c) - 2(b + 4c)$
6. $-4(a - 3b) - 3(-3a - b)$