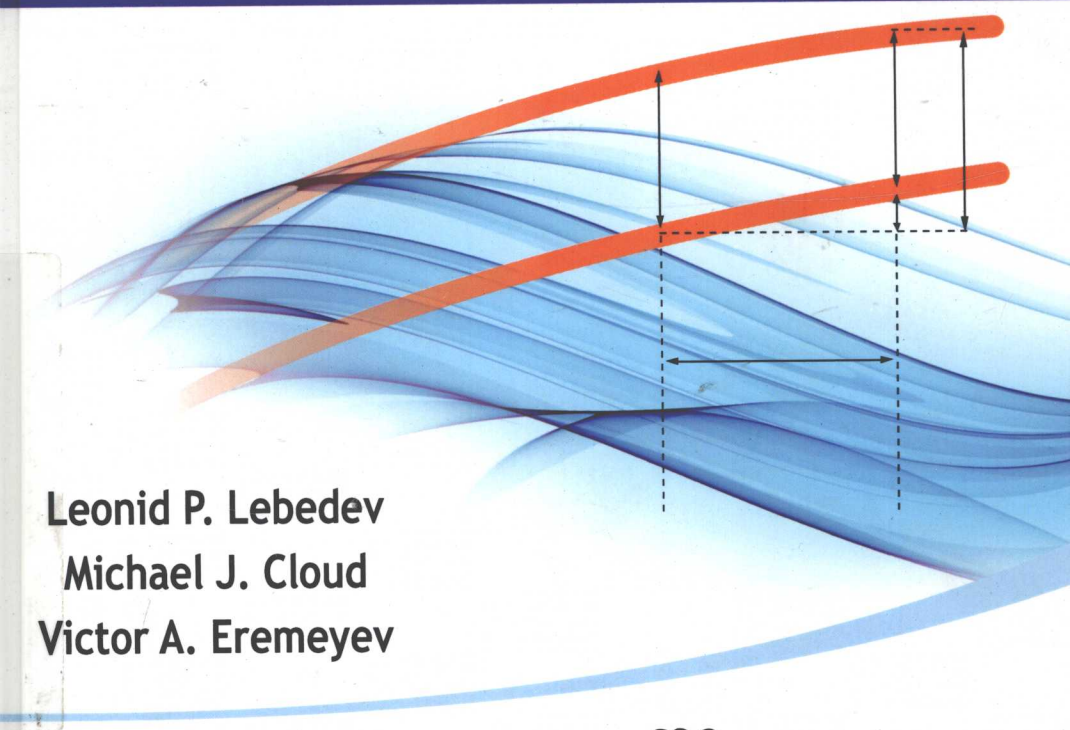
ADVANCED ENGINEERING ANALYSIS

The Calculus of Variations and Functional Analysis with Applications in Mechanics



Leonid P. Lebedev
Michael J. Cloud
Victor A. Eremeyev

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Leonid P. Lebedev
Department of Mathematics
National University of Colombia, Colombia

Michael J. Cloud
Department of Electrical and Computer Engineering,
Lawrence Technological University, USA

Victor A. Eremeyev
Institute of Mechanics, Otto von Guericke University Magdeburg, Germany
South Scientific Center of RASci
and South Federal University, Rostov on Don, Russia

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Preface

A little over half a century ago, it was said that even an ingenious person could not be an engineer unless he had nearly perfect skills with the logarithmic slide rule. The advent of the computer changed this situation crucially; at present, many young engineers have never heard of the slide rule. The computer has profoundly changed the mathematical side of the engineering profession. Symbolic manipulation programs can calculate integrals and solve ordinary differential equations better and faster than professional mathematicians can. Computers also provide solutions to differential equations in numerical form. The easy availability of modern graphics packages means that many engineers prefer such approximate solutions even when exact analytical solutions are available.

Because engineering courses must provide an understanding of the fundamentals, they continue to focus on simple equations and formulas that are easy to explain and understand. Moreover, it is still true that students must develop some analytical abilities. But the practicing engineer, armed with a powerful computer and sophisticated canned programs, employs models of processes and objects that are mathematically well beyond the traditional engineering background. The mathematical methods used by engineers have become quite sophisticated. With insufficient base knowledge to understand these methods, engineers may come to believe that the computer is capable of solving any problem. Worse yet, they may decide to accept nearly any formal result provided by a computer as long as it was generated by a program of a known trademark.

But mathematical methods are restricted. Certain problems may appear to fall within the nominal solution capabilities of a computer program and yet lie well beyond those capabilities. Nowadays, the properties of sophisticated models and numerical methods are explained using terminology

from functional analysis and the modern theory of differential equations. Without understanding terms such as "weak solution" and "Sobolev space", one cannot grasp a modern convergence proof or follow a rigorous discussion of the restrictions placed on a mathematical model. Unfortunately, the mathematical portion of the engineering curriculum remains preoccupied with 19th century topics, even omitting the calculus of variations and other classical subjects. It is, nevertheless, increasingly more important for the engineer to understand the theoretical underpinning of his instrumentation than to have an ability to calculate integrals or generate series solutions of differential equations.

The present text offers rigorous insight and will enable an engineer to communicate effectively with the mathematicians who develop models and methods for machine computation. It should prove useful to those who wish to employ modern mathematical methods with some depth of understanding.

The book constitutes a substantial revision and extension of the earlier book *The Calculus of Variations and Functional Analysis*, written by the first two authors. A new chapter (Chapter 2) provides applications of the calculus of variations to nonstandard problems in mechanics. Numerous exercises (most with extensive hints) have been added throughout.

The numbering system is as follows. All definitions, theorems, corollaries, lemmas, remarks, conventions, and examples are numbered consecutively by chapter (thus Definition 1.7 is followed by Lemma 1.8). Equations are numbered independently, again by chapter.

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Leonid P. Lebedev
Department of Mathematics, National University of Colombia, Colombia

Michael J. Cloud
Department of Electrical and Computer Engineering, Lawrence Technological University, USA

Victor A. Eremeyev
Institute of Mechanics, Otto von Guericke University Magdeburg, Germany
South Scientific Center of RASci and South Federal University, Rostov on Don, Russia

Contents

Chapter 1

Basic Calculus of Variations

1.1 Introduction

Optimization is a universal goal. Students would like to learn more, receive better grades, and have more free time; professors (at least some of them) would like to give better lectures, see students learn more, receive higher pay, and have more free time. These are the optimization problems of real life. In mathematics, optimization makes sense only when formulated in terms of a function $f(x)$ or other expression. One then seeks the minimum value of the expression. (It suffices to discuss minimization because maximizing f is equivalent to minimizing $-f$.)

This book treats the minimization of *functionals*. The notion of functional generalizes that of function. Although the process of generalization does yield results of greater generality, as a rule the results are not sharper in particular cases. So to understand what can be expected from the calculus of variations, we should review the minimization of ordinary functions. All quantities will be assumed sufficiently differentiable for the purpose at hand. Let us recall some terminology for the one-variable case $y = f(x)$.

Definition 1.1. The function $f(x)$ has a *local minimum* at a point x_0 if there is a neighborhood $(x_0 - d, x_0 + d)$ in which $f(x) \geq f(x_0)$. We call x_0 the *global minimum* of $f(x)$ on $[a, b]$ if $f(x) \geq f(x_0)$ holds for all $x \in [a, b]$.

The necessary condition for a differentiable function $f(x)$ to have a local minimum at x_0 is

$$f'(x_0) = 0. \tag{1.1}$$

A simple and convenient sufficient condition is

$$f''(x_0) > 0. \tag{1.2}$$

Unfortunately, no available criterion for a local minimum is both sufficient and necessary. So the approach is to solve (1.1) for possible points of local minimum of $f(x)$ and then test these using an available sufficient condition.

The global minimum on $[a, b]$ can be attained at a point of local minimum. But there are two points, a and b, where (1.1) may not hold (because the corresponding neighborhoods are one-sided) but where the global minimum may still occur. Hence given a differentiable function $f(x)$ on $[a, b]$, we first find all x_k at which $f'(x_k) = 0$. We then calculate $f(a)$, $f(b)$, and $f(x_k)$ at the x_k, and choose the global minimum. Although this method can be arranged as an algorithm suitable for machine computation, it still cannot be reduced to the solution of an equation or system of equations.

These tools are extended to multivariable functions and to more complex objects called *functionals*. A simple example of a functional is an integral whose integrand depends on an unknown function and its derivative. Since the extension of ordinary minimization methods to functionals is not straightforward, we continue to examine some notions from calculus.

A continuously differentiable function $f(x)$ obeys Lagrange's formula

$$f(x + h) - f(x) = f'(x + \theta h)h \qquad (0 \leq \theta \leq 1). \tag{1.3}$$

Continuity of f' means that

$$f'(x + \theta h) - f'(x) = r_1(x, \theta, h) \to 0 \quad \text{as } h \to 0,$$

hence

$$f(x + h) = f(x) + f'(x)h + r_1(x, \theta, h)\, h$$

where $r_1(x, \theta, h) \to 0$ as $h \to 0$. The term $r_1(x, \theta, h)\, h$ is Lagrange's form of the remainder. There is also Peano's form

$$f(x + h) = f(x) + f'(x)h + o(h), \tag{1.4}$$

which means that

$$\lim_{h \to 0} \frac{f(x + h) - f(x) - f'(x)h}{h} = 0.$$

The principal (linear in h) part of the increment of f is the *first differential* of f at x. Writing $dx = h$ we have

$$df = f'(x)\, dx. \tag{1.5}$$

"Infinitely small" quantities are *not* implied by this notation; here dx is a finite increment of x (taken sufficiently small when used for approximation).

The first differential is invariant under the change of variable $x = \varphi(s)$:

$$df = f'(x)\,dx = \frac{df(\varphi(s))}{ds}\,ds, \quad \text{where} \quad dx = \varphi'(s)\,ds.$$

Lagrange's formula extends to functions having m continuous derivatives in some neighborhood of x. The extension for $x + h$ lying in the neighborhood is Taylor's formula:

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \cdots + \frac{1}{(m-1)!}f^{(m-1)}(x)h^{m-1} + \frac{1}{m!}f^{(m)}(x+\theta h)h^m \qquad (0 \le \theta \le 1). \tag{1.6}$$

Continuity of $f^{(m)}$ at x yields

$$f^{(m)}(x+\theta h) - f^{(m)}(x) = r_m(x,\theta,h) \to 0 \quad \text{as } h \to 0,$$

hence Taylor's formula becomes

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \cdots + \frac{1}{m!}f^{(m)}(x)h^m + \frac{1}{m!}r_m(x,\theta,h)h^m$$

with remainder in Lagrange form. The dependence of the remainder on the parameters is suppressed in Peano's form

$$f(x+h) = f(x) + f'(x)h + \frac{1}{2!}f''(x)h^2 + \cdots + \frac{1}{m!}f^{(m)}(x)h^m + o(h^m). \tag{1.7}$$

The conditions of minimum (1.1)–(1.2) can be derived via Taylor's formula for a twice continuously differentiable function having

$$f(x+h) - f(x) = f'(x)h + \frac{1}{2}f''(x)h^2 + o(h^2). \tag{1.8}$$

Indeed $f(x+h) - f(x) \ge 0$ if x is a local minimum. The right side has the form $ah + bh^2 + o(h^2)$. If $a = f'(x) \neq 0$, for example when $a < 0$, then for $h < h_0$ with sufficiently small h_0 the sign of $f(x+h) - f(x)$ is determined by that of ah; hence for $0 < h < h_0$ we have $f(x+h) - f(x) < 0$, which contradicts the assertion that x minimizes f. The case $a > 0$ is similar, resulting in the necessary condition (1.1). The increment formula gives

$$f(x+h) - f(x) = \frac{1}{2}f''(x)h^2 + o(h^2).$$

The term $f''(x)h^2$ defines the value of the right side when h is sufficiently close to 0, hence when $f''(x) > 0$ we see that for sufficiently small $|h| \neq 0$

$$f(x+h) - f(x) > 0.$$

So (1.2) is sufficient for x to be a minimum point of f.

A function in n variables

Consider the minimization of a function $y = f(\mathbf{x})$ with $\mathbf{x} = (x_1, \ldots, x_n)$. More cannot be expected from this theory than from the theory of functions in a single variable.

Definition 1.2. A function $f(\mathbf{x})$ has a *global minimum* at the point $\mathbf{x}^*$ if the inequality

$$f(\mathbf{x}^*) \leq f(\mathbf{x}^* + \mathbf{h}) \tag{1.9}$$

holds for all nonzero $\mathbf{h} = (h_1, \ldots, h_n) \in \mathbb{R}^n$. The point $\mathbf{x}^*$ is a *local minimum* if there exists $\rho > 0$ such that (1.9) holds whenever $\|\mathbf{h}\| = (h_1^2 + \cdots + h_n^2)^{1/2} < \rho$.

Let $\mathbf{x}^*$ be a minimum point of a continuously differentiable function $f(\mathbf{x})$. Then $f(x_1, x_2^*, \ldots, x_n^*)$ is a function in one variable x_1 and takes its minimum at x_1^*. It follows that $\partial f/\partial x_1 = 0$ at $x_1 = x_1^*$. Similarly, the rest of the partial derivatives of f are zero at $\mathbf{x}^*$:

$$\left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^*} = 0, \qquad i = 1, \ldots, n. \tag{1.10}$$

This is a necessary condition of minimum for a continuously differentiable function in n variables at the point $\mathbf{x}^*$.

To get sufficient conditions we must extend Taylor's formula. Let $f(\mathbf{x})$ possess all continuous derivatives up to order $m \geq 2$ in a ball centered at point $\mathbf{x}$, and suppose $\mathbf{x} + \mathbf{h}$ lies in this ball. Fixing these, we apply (1.7) to $f(\mathbf{x} + t\mathbf{h})$ and get Taylor's formula in the variable t:

$$f(\mathbf{x}+t\mathbf{h}) = f(\mathbf{x}) + \left.\frac{df(\mathbf{x}+t\mathbf{h})}{dt}\right|_{t=0} t + \frac{1}{2!}\left.\frac{d^2 f(\mathbf{x}+t\mathbf{h})}{dt^2}\right|_{t=0} t^2 + \cdots + \frac{1}{m!}\left.\frac{d^m f(\mathbf{x}+t\mathbf{h})}{dt^m}\right|_{t=0} t^m + o(t^m).$$

The remainder term is for the case when $t \to 0$. From this equality for sufficiently small t, the general Taylor formula can be derived.

The minimization problem for $f(\mathbf{x})$ is studied using only the first two terms of this formula:

$$f(\mathbf{x}+t\mathbf{h}) = f(\mathbf{x}) + \left.\frac{df(\mathbf{x}+t\mathbf{h})}{dt}\right|_{t=0} t + \frac{1}{2!}\left.\frac{d^2 f(\mathbf{x}+t\mathbf{h})}{dt^2}\right|_{t=0} t^2 + o(t^2). \tag{1.11}$$

We calculate $df(\mathbf{x} + t\mathbf{h})/dt$ as a derivative of a composite function:

$$\left.\frac{df(\mathbf{x}+t\mathbf{h})}{dt}\right|_{t=0} = \frac{\partial f(\mathbf{x})}{\partial x_1} h_1 + \frac{\partial f(\mathbf{x})}{\partial x_2} h_2 + \cdots + \frac{\partial f(\mathbf{x})}{\partial x_n} h_n.$$

The first differential is defined as

$$df = \frac{\partial f(\mathbf{x})}{\partial x_1}\,dx_1 + \frac{\partial f(\mathbf{x})}{\partial x_2}\,dx_2 + \cdots + \frac{\partial f(\mathbf{x})}{\partial x_n}\,dx_n. \tag{1.12}$$

The next term,

$$\left.\frac{d^2 f(\mathbf{x}+t\mathbf{h})}{dt^2}\right|_{t=0} = \sum_{i,j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i\,\partial x_j} h_i h_j,$$

defines the second differential of f:

$$d^2 f = \sum_{i,j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i\,\partial x_j}\,dx_i\,dx_j. \tag{1.13}$$

Taylor's formula of the second order becomes

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} h_i + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} h_i h_j + o(\|\mathbf{h}\|^2). \tag{1.14}$$

The necessary condition for a minimum, $df = 0$, follows from (1.11) or (1.10). By (1.11), the condition

$$\left.\frac{d^2 f(\mathbf{x}+t\mathbf{h})}{dt^2}\right|_{t=0} > 0 \text{ for any sufficiently small } \|\mathbf{h}\|$$

suffices for $\mathbf{x}$ to minimize f. The corresponding quadratic form in the variables h_i is

$$\frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f(\mathbf{x})}{\partial x_i \partial x_j} h_i h_j = \frac{1}{2} \begin{pmatrix} h_1 & \cdots & h_n \end{pmatrix} \begin{pmatrix} \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1^2} & \cdots & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_1 x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f(\mathbf{x})}{\partial x_n x_1} & \cdots & \dfrac{\partial^2 f(\mathbf{x})}{\partial x_n^2} \end{pmatrix} \begin{pmatrix} h_1 \\ \vdots \\ h_n \end{pmatrix}.$$

The $n \times n$ *Hessian matrix* is symmetric under our smoothness assumptions on f. Positive definiteness of the quadratic form can be verified via Sylvester's criterion.

The problem of global minimum for a function in many variables on a closed domain Ω is more complicated than the corresponding problem for a function in one variable. Indeed, the set of points satisfying (1.10) can be infinite for a multivariable function. Trouble also arises concerning the domain boundary $\partial\Omega$: since it is no longer a finite set (unlike $\{a, b\}$) we must also solve the problem of minimum on $\partial\Omega$, and the structure of such a set can be complicated. The algorithm for finding a point of global minimum

of a function $f(\mathbf{x})$ cannot be described in several phrases; it depends on the structure of both the function and the domain.

Issues connected with the boundary can be avoided by considering the problem of global minimum of a function on an open domain. We will take this approach when treating the calculus of variations. Although analogous problems with closed domains arise in applications, the difficulties are so great that no general results are applicable to many problems. One must investigate each such problem separately.

Constraints of the form

$$g_i(\mathbf{x}) = 0, \qquad i = 1, \ldots, m, \tag{1.15}$$

permit reduction of constrained minimization to an unconstrained problem provided we can solve (1.15) and get

$$x_k = \psi_k(x_1, \ldots, x_{n-m}), \qquad k = n - m + 1, \ldots, n.$$

Substitution into $f(\mathbf{x})$ would yield an ordinary unconstrained minimization problem for a function in $n - m$ variables

$$f(x_1, \ldots, x_{n-m}, \ldots, \psi_n(x_1, \ldots, x_{n-m})).$$

The resulting system of equations is nonlinear in general. This situation can be circumvented by the use of Lagrange multipliers. The method proceeds with formation of the *Lagrangian function*

$$\mathcal{L}(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_m) = f(\mathbf{x}) + \sum_{j=1}^{m} \lambda_j g_j(\mathbf{x}), \tag{1.16}$$

by which the constraints g_j are adjoined to f. Then the x_i and λ_i are all treated as independent, unconstrained variables. The resulting necessary conditions form a system of $n + m$ equations in the $n + m$ unknowns x_i, λ_j:

$$\begin{aligned} &\frac{\partial f(\mathbf{x})}{\partial x_i} + \sum_{j=1}^{m} \lambda_j \frac{\partial g_j(\mathbf{x})}{\partial x_i} = 0, \qquad i = 1, \ldots, n, \\ &g_j(\mathbf{x}) = 0, \qquad j = 1, \ldots, m. \end{aligned} \tag{1.17}$$

Functionals

The kind of dependence in which a real number corresponds to another (or to a finite set) is not enough to describe many natural processes. Areas such as physics and biology spawn formulations not amenable to such description. Consider the deformations of an airplane in flight. At some

point near an engine, the deformation is not merely a function of the force produced by the engine — it also depends on the other engines, air resistance, and passenger positions and movements (hence the admonition that everyone remain seated during potentially dangerous parts of the flight). In general, many real processes in a body are described by the dependence of the displacement field (e.g., the field of strains, stresses, heat, voltage) on other fields (e.g., loads, heat radiation) in the same body. Each field is described by one or more functions, so the dependence is that of a function uniquely defined by a set of other functions acting as whole objects (arguments). A dependence of this type, provided we specify the classes to which all functions belong, is called an *operator* (or *map*, or sometimes just a "function" again). Problems of finding such dependences are often formulated as boundary or initial-boundary value problems for partial differential equations. These and their analysis form the main content of any course in a particular science. Since a full description of any process is complex, we usually work with simplified models that retain only essential features. However, even these can be quite challenging when we seek solutions.

Humans often try to optimize their actions through an intuitive — not mathematical — approach to fuzzily-posed problems on minimization or maximization. This is because our nature reflects the laws of nature in total. In physics there are quantities, like energy and enthalpy, whose values in the state of equilibrium or real motion are minimal or maximal in comparison with other "nearby admissible" states. Younger sciences like mathematical biology attempt to follow suit: when possible they seek to describe system behavior through the states of certain fields of parameters, on which functions of energy type attain maxima or minima. The energy of a system (e.g., body or set of interacting bodies) is characterized by a number which depends on the fields of parameters inside the system. Thus the dependence described by quantities of energy type is such that *a numerical value E is uniquely defined by the distribution of fields of parameters characterizing the system.* We call this sort of dependence a *functional.* Of course, in mathematics we must also specify the classes to which the above fields may belong. The notion of functional generalizes that of function so that the minimization problem remains sensible. Hence we come to the object of investigation of our main subject: the calculus of variations. In actuality we shall consider a somewhat restricted class of functionals. (Optimization of general functionals belongs to *mathematical programming*, a younger science that contains the calculus of variations — a subject some 300 years old — as a special case.) In the calculus of variations we min-