

Applied Mathematics

**Fourth
Edition**

J. David Logan

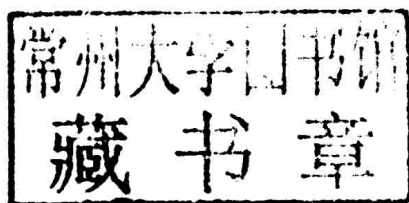
WILEY

Applied Mathematics

Fourth Edition

J. David Logan

Willa Cather Professor of Mathematics
University of Nebraska, Lincoln
Department of Mathematics
Lincoln, NE



WILEY

Cover Design: Wiley

Cover Illustration: © iStockphoto.com/hakusan

Copyright © 2013 by John Wiley & Sons, Inc. All rights reserved.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.

Published simultaneously in Canada.

No part of this publication may be reproduced, stored in a retrieval system or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, scanning or otherwise, except as permitted under Section 107 or 108 of the 1976 United States Copyright Act, without either the prior written permission of the Publisher, or authorization through payment of the appropriate per-copy fee to the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, (978) 750-8400, fax (978) 750-4470, or on the web at www.copyright.com. Requests to the Publisher for permission should be addressed to the Permissions Department, John Wiley & Sons, Inc., 111 River Street, Hoboken, NJ 07030, (201) 748-6011, fax (201) 748-6008, or online at <http://www.wiley.com/go/permission>.

Limit of Liability/Disclaimer of Warranty: While the publisher and author have used their best efforts in preparing this book, they make no representation or warranties with respect to the accuracy or completeness of the contents of this book and specifically disclaim any implied warranties of merchantability or fitness for a particular purpose. No warranty may be created or extended by sales representatives or written sales materials. The advice and strategies contained herein may not be suitable for your situation. You should consult with a professional where appropriate. Neither the publisher nor author shall be liable for any loss of profit or any other commercial damages, including but not limited to special, incidental, consequential, or other damages.

For general information on our other products and services please contact our Customer Care Department within the United States at (800) 762-2974, outside the United States at (317) 572-3993 or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print, however, may not be available in electronic formats. For more information about Wiley products, visit our web site at www.wiley.com.

Library of Congress Cataloging-in-Publication Data:

Logan, J. David (John David)

Applied mathematics / J. David Logan. — 4th ed.
pages cm

Includes bibliographical references and index.

ISBN 978-1-118-47580-5 (hardback) — ISBN 978-1-118-51492-4 — ISBN 978-1-118-51493-1 — ISBN 978-1-118-51490-0 1. Mathematics—Textbooks. I. Title.

QA37.3.L64 2013

510—dc23

2013001305

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

Applied Mathematics

To my parents

George Edd Logan (1919-1996)

Dorothy Elizabeth Wyatt Logan (1923-2009)

Preface

The fourth edition of *Applied Mathematics* shares the same goals, philosophy, and style as its predecessors—to introduce key ideas about mathematical methods and modeling, along with the important tools, to mature seniors and graduate students in mathematics, science, and engineering. The emphasis is on how mathematics interrelates with the applied and natural sciences. Prerequisites include a good command of concepts and techniques of calculus, and sophomore-level courses in differential equations and matrices; a genuine interest in applications in some area of science or engineering is a must.

Readers should understand the limited scope of this text. Being a broad introduction to the methods of applied mathematics, it cannot cover every topic in depth. Indeed, each chapter could be expanded into a one, or even several, full-length books. In fact, readers can find elementary books on all of the topics; some of these are cited in the references at the end of the chapters. Secondly, readers should understand the mathematical level of the text. Some books on applied mathematics take a highly practical approach and ignore technical mathematical issues completely, while others take a purely theoretical approach; both of these approaches are valuable and part of the overall body of applied mathematics. Here, we seek a middle ground by providing the physical basis and motivation for the ideas and methods, and we also give a glimpse of deeper mathematical ideas.

There are major changes in the fourth edition. The material has been rearranged and basically divided into two parts. Chapters 1 through 5 involve models leading to ordinary differential equations and integral equations, while Chapters 6 through 8 focus on partial differential equations and their applications. Motivated by problems in the biological sciences where quantitative methods are becoming central, Chapter 9 deals with discrete-time models,

which include some material on random processes. Sections reviewing elementary methods for solving systems of ordinary differential equations have been added in Chapters 1 and 2. Many additional examples and figures are included in this edition, and several new exercises appear throughout. Some exercises from the last edition have been revised for better clarity, and many new exercises are included. The length of the text has expanded over 160 pages. The Table of Contents details the specific topics covered.

Note that equations are numbered within sections. Thus, equation label (3.2) refers to the second numbered equation in Section 3 of the current chapter.

My colleagues in Lincoln, who have often used the text in our core sequence in applied mathematics, deserve special thanks. Glenn Ledder, Richard Rebarber, and Tom Shores have provided me with an extensive *errata*, and they supplied several exercises from graduate qualifying examinations, homework, and course exams. Former students Bill Wolesensky and Kevin TeBeest read parts of the earlier manuscripts and both were often a sounding board for suggestions. I am extremely humbled and grateful to those who used earlier editions of the book and helped establish it as one of the basic textbooks in the area; many have generously given me corrections and suggestions, and many of the typographical errors from the third edition have been resolved. Because of the extensive revision, some new ones, but hopefully not many, have no doubt appeared. I welcome suggestions, comments, and corrections, and contact information is on the book's website: <http://www.unl.edu/~jlogan1/applied-math.htm>. Solutions to some of the exercises and an *errata* will appear when they become available.

My editor at Wiley, Susanne Steitz-Filler, along with Jackie Palmieri, deserves praise for her continued enthusiasm about this new revision and her skill in making it an efficient, painless process. Finally, my wife, Tess, has been a constant source of support for my research, teaching, and writing, and I again take this opportunity to publicly express my appreciation for her encouragement and affection.

Suggestions for use of the text. The full text cannot be covered in a two-semester, 3-credit course, but there is a lot of flexibility built into the text. There is significant independence among chapters, enabling instructors to design special one- or two-semester courses in applied mathematics that meet their specific needs.

Portions of Chapters 1 through 5 can form the basis of a one-semester course involving differential and integral equations and the basic core of applied mathematics. Chapter 4 on the calculus of variations is essentially independent from the others, so it need not be covered. If students have a strong background in differential equations, then only small portions of Chapters 1 and 2 need to be covered.

A second semester, focused around partial differential equations, could cover Chapters 6, 7, and 8. Students have the flexibility to take the second semester, as is often done at the University of Nebraska, without having the first, provided small portions of Chapter 5 on Fourier-type expansions is covered.

Chapter 9, like Chapter 3, is independent from the rest of the book and can be covered at any time.

The text, and its translations, have been used in several types of courses: applied mathematics, mathematical modeling, differential equations, mathematical biology, mathematical physics, and mathematical methods in chemical or mechanical engineering.

J. David Logan, Lincoln, Nebraska

April 2013

Contents

Preface	xiii
1. Dimensional Analysis and One-Dimensional Dynamics	1
1.1 Dimensional Analysis	2
1.1.1 The Program of Applied Mathematics	2
1.1.2 Dimensional Methods	5
1.1.3 The Pi Theorem	10
1.1.4 Proof of the Pi Theorem	22
1.2 Scaling	30
1.2.1 Characteristic Scales	30
1.2.2 A Chemical Reactor Problem	33
1.2.3 The Projectile Problem	36
1.3 Differential Equations	46
1.3.1 Review of Elementary Methods	47
1.3.2 Stability and Bifurcation	58
2. Two-Dimensional Dynamical Systems	77
2.1 Phase Plane Phenomena	77
2.2 Linear Systems	87
2.3 Nonlinear Systems	94
2.4 Bifurcations	103
2.5 Reaction Kinetics	112
2.5.1 The Law of Mass Action	113
2.5.2 Enzyme Kinetics	123
2.6 Pathogens	126
2.6.1 Virus Infections	127

2.6.2	Immune System Response	130
2.6.3	Epidemics in Populations	133
2.6.4	Macroparasitic Infections	139
3.	Perturbation Methods and Asymptotic Expansions	149
3.1	Regular Perturbation	150
3.1.1	Motion in a Resistive Medium	153
3.1.2	Nonlinear Oscillations	155
3.1.3	The Poincaré–Lindstedt Method	158
3.1.4	Asymptotic Analysis	160
3.2	Singular Perturbation	170
3.2.1	Algebraic Equations	170
3.2.2	Differential Equations	173
3.2.3	Boundary Layers	174
3.3	Boundary Layer Analysis	179
3.3.1	Inner and Outer Approximations	179
3.3.2	Matching	181
3.3.3	Uniform Approximations	183
3.3.4	General Procedures	186
3.4	Initial Layers	191
3.4.1	Damped Spring–Mass System	191
3.4.2	Enzyme Kinetics	195
3.5	The WKB Approximation	202
3.5.1	The Nonoscillatory Case	205
3.5.2	The Oscillatory Case	207
3.6	Asymptotic Expansion of Integrals	210
3.6.1	Laplace Integrals	211
3.6.2	Integration by Parts	214
3.6.3	Other Integrals	216
4.	Calculus of Variations	221
4.1	Variational Problems	221
4.1.1	Functionals	221
4.1.2	Examples	223
4.2	Necessary Conditions for Extrema	227
4.2.1	Normed Linear Spaces	227
4.2.2	Derivatives of Functionals	231
4.2.3	Necessary Conditions	233
4.3	The Simplest Problem	236
4.3.1	The Euler Equation	236
4.3.2	Solved Examples	239
4.3.3	First Integrals	240

4.4	Generalizations	245
4.4.1	Higher Derivatives	245
4.4.2	Several Functions	247
4.4.3	Natural Boundary Conditions	249
4.5	Hamilton's Principle	253
4.5.1	Hamilton's Equations	259
4.5.2	The Inverse Problem	262
4.6	Isoperimetric Problems	266
5.	Boundary Value Problems and Integral Equations	275
5.1	Boundary-Value Problems	277
5.2	Sturm–Liouville Problems	284
5.2.1	The Eigenvalue Problem	285
5.2.2	Eigenfunction Expansions and Bases	295
5.2.3	Best Approximation and Hilbert Spaces	302
5.3	Classical Fourier Series	310
5.4	Integral Equations	317
5.4.1	Volterra Equations	319
5.4.2	Fredholm Equations with Degenerate Kernels	325
5.4.3	Symmetric Kernels	331
5.5	Green's Functions	339
5.5.1	Inverses of Differential Operators	340
5.5.2	Physical Interpretation	342
5.5.3	Green's Function via Eigenfunctions	348
5.6	Distributions	352
5.6.1	Test Functions	352
5.6.2	Distributions	355
5.6.3	Distribution Solutions to Differential Equations	360
6.	Partial Differential Equations	365
6.1	Basic Concepts	365
6.1.1	Linearity and Superposition	370
6.2	Conservation Laws	375
6.2.1	One Dimension	375
6.2.2	Several Dimensions	378
6.2.3	Constitutive Relations	383
6.2.4	Probability and Diffusion	387
6.2.5	Boundary Conditions	390
6.3	Equilibrium Equations	397
6.3.1	Laplace's Equation	397
6.3.2	Basic Properties	401
6.4	Eigenfunction Expansions	404

6.4.1	Spectrum of the Laplacian	405
6.4.2	Evolution Problems	408
6.5	Integral Transforms	415
6.5.1	Laplace Transforms	415
6.5.2	Fourier Transforms	423
6.6	Stability of Solutions	435
6.6.1	Reaction–Diffusion Equations	435
6.6.2	Pattern Formation	437
6.7	Distributions	443
6.7.1	Elliptic Problems	443
6.7.2	Fourier Transforms of Distributions	448
6.7.3	Diffusion Problems	449
7.	Wave Phenomena	457
7.1	Waves	457
7.1.1	The Advection Equation	463
7.2	Nonlinear Waves	470
7.2.1	Nonlinear Advection	470
7.2.2	Traveling Wave Solutions	477
7.2.3	Conservation Laws	483
7.3	Quasi-linear Equations	488
7.3.1	Age-Structured Populations	492
7.4	The Wave Equation	497
7.4.1	The Acoustic Approximation	497
7.4.2	Solutions to the Wave Equation	501
7.4.3	Scattering and Inverse Problems	507
7.4.4	The Schrödinger Equation	510
8.	Mathematical Models of Continua	523
8.1	Kinematics and Mass Conservation	524
8.1.1	Description of Flow	524
8.1.2	Mass Conservation	530
8.2	Momentum and Energy	534
8.2.1	Momentum Conservation	534
8.2.2	Stress Waves in Solids	538
8.2.3	Thermodynamics and Energy Conservation	545
8.3	Gas Dynamics	551
8.3.1	Riemann’s Method	551
8.3.2	Rankine–Hugoniot Conditions	557
8.4	Fluid Motions in \mathbb{R}^3	560
8.4.1	Kinematics	560
8.4.2	Dynamics	567

8.4.3	Energy and Constitutive Theory	574
9.	Discrete Models	585
9.1	One-Dimensional Models	586
9.1.1	Linear and Nonlinear Models	586
9.1.2	Equilibria, Stability, and Chaos	591
9.2	Systems of Difference Equations	599
9.2.1	Linear Models	599
9.2.2	Nonlinear Interactions	610
9.3	Stochastic Models	619
9.3.1	Elementary Probability	619
9.3.2	Stochastic Processes	626
9.3.3	Environmental and Demographic Models	630
9.4	Probability-Based Models	636
9.4.1	Markov Processes	636
9.4.2	Random Walks	642
9.4.3	The Poisson Process	647
Index	653

