MATHEMATICAL METHODS FOR PHYSICS

H. W. Wyld

Mathematical Methods for Physics

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FOREWORD

Everyone concerned with the teaching of physics at the advanced undergraduate or graduate level is aware of the continuing need for a modernization and reorganization of the basic course material. Despite the existence today of many good textbooks in these areas, there is always an appreciable time-lag in the incorporation of new viewpoints and techniques which result from the most recent developments in physics research. Typically these changes in concepts and material take place first in the personal lecture notes of some of those who teach graduate courses. Eventually, printed notes may appear, and some fraction of such notes evolve into textbooks or monographs. But much of this fresh material remains available only to a very limited audience, to the detriment of all. Our series aims to fill this gap in the literature of physics by presenting occasional volumes with a contemporary approach to the classical topics of physics at the advanced undergraduate and graduate level. Clarity and soundness of treatment will, we hope, mark these volumes, as well as the freshness of the approach.

Another area in which the series hopes to make a contribution is by presenting useful supplementing material of well-defined scope. This may take the form of a survey of relevant mathematical principles, or a collection of reprints of basic papers in a field. Here the aim is to provide the instructor with added flexibility through the use of supplements at relatively low cost.

The scope of both the Lecture Notes and Supplements is somewhat different from the FRONTIERS IN PHYSICS Series. In spite of wide variations from institution to institution as to what comprises the basic graduate course program, there is a widely accepted group of "bread and butter" courses that deal with the classic topics in physics. These include: mathematical methods of physics, electromagnetic theory, advanced dynamics, quantum mechanics, statistical mechanics, and frequently nuclear physics and/or solid-state physics.

It is chiefly these areas that will be covered by the present series. The listing is perhaps best described as including all advanced undergraduate and graduate courses which are at a level below seminar courses dealing entirely with current research topics.

The above words were written in 1962 in collaboration with David Jackson who served as co-editor of this series during its first decade. They serve equally well as a Foreword for the present volume, which offers to the physics or engineering student a lucidly organized readable short text from which he or she can acquire much of the important mathematical knowledge necessary for a professional career. Professor Wyld's book possesses the further virtue of explaining in some detail the physics which underlies the mathematical problems considered therein, so that it provides an opportunity for students to learn more about physics, as well as many of the essential mathematical methods of physics. H. W. Wyld has made a number of significant contributions to theoretical physics, and is noted for the clarity of his lectures and his writing. It is a pleasure to welcome him as a contributor to this series.

David Pines

PREFACE

This book is a written version of the lecture course I have given over a number of years to first-year graduate students at the University of Illinois on the subject of mathematical methods for physics. The course (and the book) are intended to provide the students with the basic mathematical background which they will need to perform typical calculations in classical and quantum physics. The level is intermediate; the usual undergraduate course in advanced calculus should be an adequate prerequisite and would even provide some overlap (e.g. Fourier series) with the subjects covered in the present work. The treatment is limited to certain standard topics in classical analysis; no attempt is made to cover the method of characteristics, Hilbert space, or group theoretical methods. What I have tried to do is provide a short readable textbook from which the average physics or engineering student can learn the most important mathematical tools he will need in his professional career. The physics which lies behind the mathematical problems is all explained in some detail, so that the treatment should be intelligible also to pure mathematicians and might even provide an introduction to some of the advanced texts by mathematicians on the subject.

The mathematical methods sequence, as presently constituted at the University of Illinois, consists of three half-semester courses, i.e., all together 3/4 of an academic year. Together with a fourth half-semester course in classical mechanics, these courses provide a basis for more advanced work in electrodynamics, quantum mechanics, particle, nuclear, and solid state physics. The subject matter of the three parts, intentionally kept independent, and the corresponding chapters in the present book, are:

I. Homogeneous Boundary Value Problems and Special Functions. 1-6

The low level of mathematical rigor which is customarily found in the writing of physicists will also be found in the present work. I feel that students seriously concerned with rigor should consult the mathematicians. I have, however, attempted to give, at appropriate spots, page references to works in which rigorous mathematical proofs and accurately worded theorems can be found.

Finally, I want to record here my great debt to Mary Ostendorf for the excellent job she did in typing the manuscript.

H. W. WYLD

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HOMOGENEOUS BOUNDARY VALUE PROBLEMS AND SPECIAL FUNCTIONS

CHAPTER 1 THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

1.1 INTRODUCTION

A large fraction of classical, and also quantum, physics uses a common type of mathematics. Certain partial differential equations occur over and over again in different fields. The methods of solution of these equations and the special functions which arise are thus generally useful tools which should be known to all physicists. The purpose of this book is to provide a guide to the study of this part of mathematics and to show how it is used in various physical applications.

Similar courses are offered in mathematics departments. There, one is usually concerned with the rigorous logical development of the mathematics. The student interested in such matters should and must go to the mathematicians. Here we will minimize the rigor and concentrate on a rough and ready approach to applications.

We start by reviewing the physical basis of the various equations we wish to solve.

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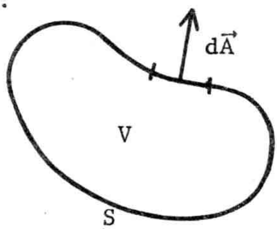
1.2 HEAT CONDUCTION AND DIFFUSION

The flow of heat through a medium can be described by a flux vector $\vec{F}$, whose direction gives the direction of the heat flow and whose magnitude gives the magnitude of the heat flow in cal./cm.^{2}/sec. This vector $\vec{F}$ is related to the gradient of the temperature T by the thermal conductivity K of the medium:

$$\vec{F} = -K \operatorname{grad} T . \tag{1.2.1}$$

We also introduce the specific heat c and the density ρ of the medium.

In terms of these quantities we can write two different but equal expressions for the rate of change with time t of the heat Q in a volume V:



$$\begin{aligned} \frac{dQ}{dt} = \int_V d^3x \, c\rho \frac{\partial T}{\partial t} &= -\int_S d\vec{A} \cdot \vec{F} \\ &= \int_S d\vec{A} \cdot K \operatorname{grad} T \\ &= \int_V d^3x \operatorname{div}(K \operatorname{grad} T) . \end{aligned} \tag{1.2.2}$$

Here the last step follows from the mathematical identity known as Gauss' theorem,

$$\int_S d\vec{A} \cdot \vec{F} = \int_V d^3x \operatorname{div} \vec{F}, \tag{1.2.3}$$

valid for any vector field $\vec{F}$. Since the volume V is arbitrary, we obtain from (1.2.2) the relation

$$c\,\rho\frac{\partial T}{\partial t} = \operatorname{div}(K \operatorname{grad} T), \tag{1.2.4}$$

or, if K is a constant,

$$\nabla^2 T = \frac{1}{\varkappa}\frac{\partial T}{\partial t} \tag{1.2.5}$$

with $\varkappa = K/c\rho$ and

$$\nabla^2 = \operatorname{div}\operatorname{grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\,. \tag{1.2.6}$$

A similar equation is obtained for processes involving the diffusion of particles. If $n(\vec{r},t)$ is the concentration of particles (number/cm.^{3}), the flux of particles is given by

$$\vec{F} = -C \operatorname{grad} n,$$

where C is a constant. We can then write two expressions for the rate of change with time of the number N of particles in a volume V:

$$\frac{dN}{dt} = \int_V d^3x\,\frac{\partial n}{\partial t} = -\int_S d\vec{A}\cdot\vec{F} = C\int_V d^3x \operatorname{div}(\operatorname{grad} n). \tag{1.2.7}$$

Since the volume V is arbitrary, we obtain

$$\nabla^2 n = \frac{1}{C}\frac{\partial n}{\partial t}\,. \tag{1.2.8}$$

The heat conduction equation (1.2.5), or diffusion equation (1.2.8), is a standard equation of mathematical physics. In the important special case of no time dependence, T or n independent of time, we obtain Laplace's equation:

$$\nabla^2 T = 0 \quad \text{or} \quad \nabla^2 n = 0. \tag{1.2.9}$$

For the less restrictive special case of exponential time dependence, $T(\vec{r},t) = e^{-\varkappa k^2 t} u(\vec{r})$ or $n(\vec{r},t) = e^{-Ck^2 t} u(\vec{r})$, we obtain the Helmholtz equation:

$$(\nabla^2 + k^2) u(\vec{r}) = 0. \tag{1.2.10}$$

1.3 QUANTUM MECHANICS

The potential V in Schrodinger's equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}, \tag{1.3.1}$$

makes each quantum mechanics problem a special case. If $V = 0$, we find

$$\nabla^2 \psi = -i \frac{2m}{\hbar} \frac{\partial \psi}{\partial t}, \tag{1.3.2}$$

which is the diffusion equation with an imaginary diffusion constant. If we assume an exponential time dependence,

$$\psi(\vec{r},t) = u(\vec{r})\, e^{-i\frac{E}{\hbar}t}, \quad E = \frac{\hbar^2 k^2}{2m}, \tag{1.3.3}$$

we again find the Helmholtz equation (1.2.10).