

# **MATHEMATICAL METHODS FOR PHYSICS**

H. W. Wyld

# Mathematical Methods for Physics

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## FOREWORD

Everyone concerned with the teaching of physics at the advanced undergraduate or graduate level is aware of the continuing need for a modernization and reorganization of the basic course material. Despite the existence today of many good textbooks in these areas, there is always an appreciable time-lag in the incorporation of new viewpoints and techniques which result from the most recent developments in physics research. Typically these changes in concepts and material take place first in the personal lecture notes of some of those who teach graduate courses. Eventually, printed notes may appear, and some fraction of such notes evolve into textbooks or monographs. But much of this fresh material remains available only to a very limited audience, to the detriment of all. Our series aims to fill this gap in the literature of physics by presenting occasional volumes with a contemporary approach to the classical topics of physics at the advanced undergraduate and graduate level. Clarity and soundness of treatment will, we hope, mark these volumes, as well as the freshness of the approach.

Another area in which the series hopes to make a contribution is by presenting useful supplementing material of well-defined scope. This may take the form of a survey of relevant mathematical principles, or a collection of reprints of basic papers in a field. Here the aim is to provide the instructor with added flexibility through the use of supplements at relatively low cost.

The scope of both the Lecture Notes and Supplements is somewhat different from the FRONTIERS IN PHYSICS Series. In spite of wide variations from institution to institution as to what comprises the basic graduate course program, there is a widely accepted group of "bread and butter" courses that deal with the classic topics in physics. These include: mathematical methods of physics, electromagnetic theory, advanced dynamics, quantum mechanics, statistical mechanics, and frequently nuclear physics and/or solid-state physics.

It is chiefly these areas that will be covered by the present series. The listing is perhaps best described as including all advanced undergraduate and graduate courses which are at a level below seminar courses dealing entirely with current research topics.

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The above words were written in 1962 in collaboration with David Jackson who served as co-editor of this series during its first decade. They serve equally well as a Foreword for the present volume, which offers to the physics or engineering student a lucidly organized readable short text from which he or she can acquire much of the important mathematical knowledge necessary for a professional career. Professor Wyld's book possesses the further virtue of explaining in some detail the physics which underlies the mathematical problems considered therein, so that it provides an opportunity for students to learn more about physics, as well as many of the essential mathematical methods of physics. H. W. Wyld has made a number of significant contributions to theoretical physics, and is noted for the clarity of his lectures and his writing. It is a pleasure to welcome him as a contributor to this series.

David Pines

## PREFACE

This book is a written version of the lecture course I have given over a number of years to first-year graduate students at the University of Illinois on the subject of mathematical methods for physics. The course (and the book) are intended to provide the students with the basic mathematical background which they will need to perform typical calculations in classical and quantum physics. The level is intermediate; the usual undergraduate course in advanced calculus should be an adequate prerequisite and would even provide some overlap (e.g. Fourier series) with the subjects covered in the present work. The treatment is limited to certain standard topics in classical analysis; no attempt is made to cover the method of characteristics, Hilbert space, or group theoretical methods. What I have tried to do is provide a short readable textbook from which the average physics or engineering student can learn the most important mathematical tools he will need in his professional career. The physics which lies behind the mathematical problems is all explained in some detail, so that the treatment should be intelligible also to pure mathematicians and might even provide an introduction to some of the advanced texts by mathematicians on the subject.

The mathematical methods sequence, as presently constituted at the University of Illinois, consists of three half-semester courses, i. e., all together  $3/4$  of an academic year. Together with a fourth half-semester course in classical mechanics, these courses provide a basis for more advanced work in electrodynamics, quantum mechanics, particle, nuclear, and solid state physics. The subject matter of the three parts, intentionally kept independent, and the corresponding chapters in the present book, are:

- I. Homogeneous Boundary Value Problems and Special Functions. 1-6

II. Inhomogeneous Problems, Green's Functions, and Integral Equations. 7-9

III. Complex Variable Techniques. 10-14

The low level of mathematical rigor which is customarily found in the writing of physicists will also be found in the present work. I feel that students seriously concerned with rigor should consult the mathematicians. I have, however, attempted to give, at appropriate spots, page references to works in which rigorous mathematical proofs and accurately worded theorems can be found.

Finally, I want to record here my great debt to Mary Ostendorf for the excellent job she did in typing the manuscript.

H. W. WYLD

## CONTENTS

Editor's Foreword	xiii
Preface	xv
<b>Part I HOMOGENEOUS BOUNDARY VALUE PROBLEMS AND SPECIAL FUNCTIONS</b>	
<b>1. THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS</b>	<b>1</b>
1.1 Introduction	1
1.2 Heat Conduction and Diffusion	2
1.3 Quantum Mechanics	4
1.4 Waves on Strings and Membranes	5
1.5 Hydrodynamics and Aerodynamics	7
1.6 Acoustic Waves in a Compressible Fluid	9
1.7 Irrotational Flow in an Incompressible Fluid	11
1.8 Electrodynamics	14
a) Time Independent Phenomena	15
b) Vacuum Equations	18
c) General Case	18
1.9 Summary	21
Problems	23
<b>2. SEPARATION OF VARIABLES AND ORDINARY DIFFERENTIAL EQUATIONS</b>	<b>28</b>
2.1 Introduction	28
2.2 Separation of Variables	29
2.3 Rectangular Coordinates $(x, y, z)$	30
2.4 Cylindrical Coordinates $(r, \theta, z)$	32
2.5 Spherical Coordinates $(r, \theta, \phi)$	34
2.6 Series Solutions of Ordinary Differential Equations: Preliminaries	38
2.7 Expansion About a Regular Singular Point	43
2.8 Sturm Liouville Eigenvalue Problem	49
2.9 Fourier Series and Integrals	58
2.10 Numerical Solution of Ordinary Differential Equations	61
Problems	67



3.	SPHERICAL HARMONICS AND APPLICATIONS	74
3.1	Introduction	74
3.2	Series Solution of Legendre's Equation - Legendre Polynomials	75
3.3	Properties of Legendre Polynomials	81
3.4	The Second Solution $Q_\ell(x)$ of Legendre's Equation	88
3.5	Associated Legendre Polynomials	92
3.6	Spherical Harmonics	96
3.7	The Spherical Harmonics Addition Theorem	99
3.8	Multipole Expansions	103
3.9	Laplace's Equation in Spherical Coordinates	108
	1) Interior Problem	110
	2) Exterior Problem	112
	3) Region Between Two Spheres	112
3.10	Conducting Sphere in a Uniform External Electric Field	113
3.11	Flow of an Incompressible Fluid Around a Spherical Obstacle	115
	Problems	118
4.	BESSEL FUNCTIONS AND APPLICATIONS	122
4.1	Introduction	122
4.2	Series Solutions of Bessel's Equation; Bessel Functions	123
4.3	Neumann Functions	126
4.4	Small Argument and Asymptotic Expansions	132
4.5	Bessel Functions of Imaginary Argument	134
4.6	Laplace's Equation in Cylindrical Coordinates	135
4.7	Interior of a Cylinder of Finite Length	136
4.8	The Sturm Liouville Eigenvalue Problem and Expansion Theorem	139
4.9	Interior of a Cylinder of Finite Length - Continued	141
4.10	Exterior of an Infinitely Long Cylinder	144
4.11	Cylinder in an External Field	145
4.12	Space Between Two Infinite Planes	147
4.13	Fourier Bessel Transforms	149
4.14	Space Between Two Infinite Planes - Continued Problems	151 152
5.	NORMAL MODE EIGENVALUE PROBLEMS	154
5.1	Introduction	154
5.2	Reduction of the Diffusion Equation and Wave Equation to an Eigenvalue Problem	155
5.3	The Vibrating String	161
5.4	The Vibrating Drumhead	162

5.5	Heat Conduction in a Cylinder of Finite Length	166
5.6	Particle in a Cylindrical Box (Quantum Mechanics)	168
5.7	Normal Modes of an Acoustic Resonant Cavity	168
5.8	Acoustic Wave Guide	170
	Problems	174
6.	SPHERICAL BESSEL FUNCTIONS AND APPLICATIONS	177
6.1	Introduction	177
6.2	Formulas for Spherical Bessel Functions in Terms of Elementary Functions	173
6.3	Eigenvalue Problem and Expansion Theorem	184
6.4	Expansion of Plane and Spherical Waves in Spherical Coordinates	187
6.5	The Emission of Spherical Waves	192
6.6	Scattering of Waves by a Sphere	197
	Problems	204
	Part I SUMMARY OF PART I	207
	Part II INHOMOGENEOUS PROBLEMS, GREEN'S FUNCTIONS, AND INTEGRAL EQUATIONS	
7.	DIELECTRIC AND MAGNETIC MEDIA	218
7.1	Introduction	218
7.2	Macroscopic Electrostatics in the Presence of Dielectrics	219
7.3	Boundary Value Problems in Dielectrics	229
	1) Free Charge Distribution $\rho_F$ Embedded in an Infinite Uniform Dielectric with a Constant Dielectric Constant $\epsilon$	229
	2) Point Charge in Front of a Semi-infinite Dielectric	230
	3) Dielectric Sphere in a Uniform External Electric Field	233
7.4	Magnetostatics and the Multipole Expansion for the Vector Potential	235
7.5	Magnetic Media	241
7.6	Boundary Value Problems in Magnetic Media	247
	1) Uniformly Magnetized Sphere, $\vec{M}$ Given	247
	2) Magnetic Sphere in a Uniform External Magnetic Field	250
	3) Long Straight Wire Carrying Current $I$ Parallel to a Semi-infinite Slab of Material of Permeability $\mu$	253
	Problems	255

8.	GREEN'S FUNCTIONS	258
8.1	Introduction	258
8.2	Ordinary Differential Equations	259
8.3	General Theory, Various Boundary Conditions	260
	1) $u(a)$ and $u(b)$ Given	261
	2) $u(a)$ and $du(x)/dx _{x=b}$ Given	261
	3) $Au(a) + Bu'(a) = X$ Given and $Cu(b) + Du'(b) = Y$ Given	262
8.4	The Bowed Stretched String	265
8.5	Expansion of Green's Function in Eigenfunctions	269
8.6	Poisson's Equation	272
	a) $\psi(\vec{r}')$ Given on S	273
	b) $\partial\psi(\vec{r}')/\partial n'$ Given on S	274
8.7	Poisson's Equation for All Space	276
8.8	Electrostatics with Boundary Conditions on Surfaces at Finite Distances — The Image Method	276
8.9	Expansion of the Green's Function for the Interior of a Sphere in Series	279
8.10	The Helmholtz Equation — The Forced Drumhead	283
8.11	Eigenfunction Expansion of Green's Function for the Helmholtz Equation	289
8.12	The Helmholtz Equation for Infinite Regions, Radiation, and the Wave Equation; Sinusoidal Time Dependence	291
8.13	General Time Dependence	295
8.14	The Wave Equation	299
8.15	The Wave Equation for All Space, No Boundaries at Finite Distances	304
8.16	Field Due to a Point Source	311
	1) Point Source Moving with Constant Velocity, $v < c$	314
	2) Point Source Moving with Constant Velocity, $v > c$	315
8.17	The Diffusion Equation	319
8.18	The Diffusion Equation for All Space, No Boundaries at Finite Distances	322
	Problems	327
9.	INTEGRAL EQUATIONS	335
9.1	Introduction	335
9.2	Quantum Theory of Scattering	336
9.3	Types of Integral Equations	341
	1) First Kind	341
	2) Second Kind	341
	3) Volterra.	341
	4) Eigenvalue Problem	341

9.4	Integral Equations with Separable Kernels	342
9.5	Convolution Integral Equations	347
9.6	Iteration-Liouville Neumann Series	348
9.7	Numerical Solution	351
9.8	Fredholm's Formulas	356
9.9	Conditions for Validity of Fredholm's Formulas	363
9.10	Hilbert Schmidt Theory	368
	Problems	375
Part III COMPLEX VARIABLE TECHNIQUES		
10.	COMPLEX VARIABLES; BASIC THEORY	383
10.1	Introduction	383
10.2	Analytic Functions; The Cauchy-Riemann Equations	385
10.3	Power Series	394
10.4	Multivalued Functions; Cuts; Riemann Sheets	402
10.5	Contour Integrals; Cauchy's Theorem	417
10.6	Cauchy's Integral Formula	426
10.7	Taylor and Laurent Expansions	428
10.8	Analytic Continuation	435
	Problems	443
11.	EVALUATION OF INTEGRALS	448
11.1	Introduction	448
11.2	The Residue Theorem	448
11.3	Rational Functions ( $-\infty, \infty$ )	451
11.4	Exponential Factors; Jordan's Lemma	453
11.5	Integrals on the Range $(0, \infty)$	459
11.6	Angular Integrals	462
11.7	Transforming the Contour	464
11.8	Partial Fraction and Product Expansions	466
	Problems	473
12.	DISPERSION RELATIONS	475
12.1	Introduction	475
12.2	Plemelj Formulas; Dirac's Formula	476
12.3	Discontinuity Problem	479
12.4	Dispersion Relations; Spectral Representations	481
12.5	Examples	493
12.6	Integral Equations with Cauchy Kernels	499
	Problems	506
13.	SPECIAL FUNCTIONS	507
13.1	Introduction	507
13.2	The Gamma Function	508
13.3	Asymptotic Expansions; Stirling's Formula	513
13.4	The Hypergeometric Function	519
13.5	Legendre Functions	533

13.6	Bessel Functions	541
13.7	Asymptotic Expansions for Bessel Functions	552
	Problems	561
14.	INTEGRAL TRANSFORMS IN THE COMPLEX PLANE	564
14.1	Introduction	564
14.2	The Calculation of Green's Functions by Fourier Transform Methods	566
	a) The Helmholtz Equation	566
	b) The Wave Equation	570
	c) The Klein Gordon Equation	573
14.3	One-Sided Fourier Transforms; Laplace Transforms	581
14.4	Linear Differential Equations with Constant Coefficients	588
14.5	Integral Equations of Convolution Type	590
14.6	Mellin Transforms	590
14.7	Partial Differential Equations	592
14.8	The Wiener-Hopf Method	596
	1) Potential Given on Semi-Infinite Plate	597
	2) Diffraction by a Knife Edge	604
	Problems	618
	Bibliography	621
	Index	625

# HOMOGENEOUS BOUNDARY

## VALUE PROBLEMS

### AND

## SPECIAL FUNCTIONS

# CHAPTER 1 THE PARTIAL DIFFERENTIAL EQUATIONS OF MATHEMATICAL PHYSICS

## 1.1 INTRODUCTION

A large fraction of classical, and also quantum, physics uses a common type of mathematics. Certain partial differential equations occur over and over again in different fields. The methods of solution of these equations and the special functions which arise are thus generally useful tools which should be known to all physicists. The purpose of this book is to provide a guide to the study of this part of mathematics and to show how it is used in various physical applications.

Similar courses are offered in mathematics departments. There, one is usually concerned with the rigorous logical development of the mathematics. The student interested in such matters should and must go to the mathematicians. Here we will minimize the rigor and concentrate on a rough and ready approach to applications.

We start by reviewing the physical basis of the various equations we wish to solve.

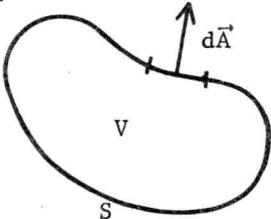
## 1.2 HEAT CONDUCTION AND DIFFUSION

The flow of heat through a medium can be described by a flux vector  $\vec{F}$ , whose direction gives the direction of the heat flow and whose magnitude gives the magnitude of the heat flow in cal./cm.<sup>2</sup>/sec. This vector  $\vec{F}$  is related to the gradient of the temperature  $T$  by the thermal conductivity  $K$  of the medium:

$$\vec{F} = -K \text{grad } T. \quad (1.2.1)$$

We also introduce the specific heat  $c$  and the density  $\rho$  of the medium.

In terms of these quantities we can write two different but equal expressions for the rate of change with time  $t$  of the heat  $Q$  in a volume  $V$ :

$$\begin{aligned} \frac{dQ}{dt} &= \int_V d^3x \, c\rho \frac{\partial T}{\partial t} = - \int_S d\vec{A} \cdot \vec{F} \\ &= \int_S d\vec{A} \cdot K \text{grad } T \\ &= \int_V d^3x \, \text{div}(K \text{grad } T). \end{aligned} \quad (1.2.2)$$


The diagram shows a closed, irregularly shaped volume labeled 'V'. The boundary of this volume is labeled 'S'. On the surface 'S', a small area element is indicated by two short line segments. From this element, a vector labeled 'dA' points outwards, representing the outward normal to the surface.

Here the last step follows from the mathematical identity known as Gauss' theorem,

$$\int_S d\vec{A} \cdot \vec{F} = \int_V d^3x \, \text{div } \vec{F}, \quad (1.2.3)$$

valid for any vector field  $\vec{F}$ . Since the volume  $V$  is arbitrary, we obtain from (1.2.2) the relation

$$c \rho \frac{\partial T}{\partial t} = \text{div}(K \text{grad } T), \quad (1.2.4)$$

or, if  $K$  is a constant,

$$\nabla^2 T = \frac{1}{\kappa} \frac{\partial T}{\partial t} \quad (1.2.5)$$

with  $\kappa = K/c\rho$  and

$$\nabla^2 = \text{div grad} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \quad (1.2.6)$$

A similar equation is obtained for processes involving the diffusion of particles. If  $n(\vec{r}, t)$  is the concentration of particles (number/cm.<sup>3</sup>), the flux of particles is given by

$$\vec{F} = -C \text{grad } n,$$

where  $C$  is a constant. We can then write two expressions for the rate of change with time of the number  $N$  of particles in a volume  $V$ :

$$\frac{dN}{dt} = \int_V d^3x \frac{\partial n}{\partial t} = - \int_S d\vec{A} \cdot \vec{F} = C \int_V d^3x \text{div}(\text{grad } n). \quad (1.2.7)$$

Since the volume  $V$  is arbitrary, we obtain

$$\nabla^2 n = \frac{1}{C} \frac{\partial n}{\partial t}. \quad (1.2.8)$$

The heat conduction equation (1.2.5), or diffusion equation (1.2.8), is a standard equation of mathematical physics. In the important special case of no time dependence,  $T$  or  $n$  independent of time, we obtain Laplace's



equation:

$$\nabla^2 T = 0 \quad \text{or} \quad \nabla^2 n = 0. \quad (1.2.9)$$

For the less restrictive special case of exponential time dependence,  $T(\vec{r}, t) = e^{-\kappa k^2 t} u(\vec{r})$  or  $n(\vec{r}, t) = e^{-Ck^2 t} u(\vec{r})$ , we obtain the Helmholtz equation:

$$(\nabla^2 + k^2)u(\vec{r}) = 0. \quad (1.2.10)$$

### 1.3 QUANTUM MECHANICS.

The potential  $V$  in Schrodinger's equation,

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi = i\hbar \frac{\partial \psi}{\partial t}, \quad (1.3.1)$$

makes each quantum mechanics problem a special case. If  $V=0$ , we find

$$\nabla^2 \psi = -i \frac{2m}{\hbar} \frac{\partial \psi}{\partial t}, \quad (1.3.2)$$

which is the diffusion equation with an imaginary diffusion constant. If we assume an exponential time dependence,

$$\psi(\vec{r}, t) = u(\vec{r}) e^{-\frac{E}{\hbar} t}, \quad E = \frac{\hbar^2 k^2}{2m}, \quad (1.3.3)$$

we again find the Helmholtz equation (1.2.10).