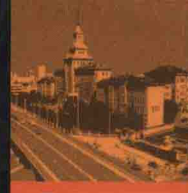

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Differential Forms—Theory and Practice

微分形式—理论与练习

[美] Weintraub, S. H. （温特劳斯） 著

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To my mother and the memory of my father

Preface

Differential forms are a powerful computational and theoretical tool. They play a central role in mathematics, in such areas as analysis on manifolds and differential geometry, and in physics as well, in such areas as electromagnetism and general relativity. In this book, we present a concrete and careful introduction to differential forms, at the upper-undergraduate or beginning graduate level, designed with the needs of both mathematicians and physicists (and other users of the theory) in mind.

On the one hand, our treatment is concrete. By that we mean that we present quite a bit of material on how to do computations with differential forms, so that the reader may effectively use them.

On the other hand, our treatment is careful. By that we mean that we present precise definitions and rigorous proofs of (almost) all of the results in this book.

We begin at the beginning, defining differential forms and showing how to manipulate them. First we show how to do algebra with them, and then we show how to find the exterior derivative $d\varphi$ of a differential form φ. We explain what differential forms really are: Roughly speaking, a k-form is a particular kind of function on k-tuples of tangent vectors. (Of course, in order to make sense of this we must first make sense of tangent vectors.) We carry on to our main goal, the Generalized Stokes's Theorem, one of the central theorems of mathematics. This theorem states:

THEOREM (*Generalized Stokes's Theorem (GST)*). *Let M be an oriented smooth k-manifold with boundary ∂M (possibly empty) and let ∂M be given the induced orientation. Let φ be a $(k-1)$-form on M with compact support. Then*

$$\int_M d\varphi = \int_{\partial M} \varphi.$$

This goal determines our path. We must develop the notion of an oriented smooth manifold and show how to integrate differential forms on these. Once we have done so, we can state and prove this theorem.

The theory of differential forms was first developed in the early twentieth century by Elie Cartan, and this theory naturally led to de Rham cohomology, which we consider in our last chapter.

One thing we call the reader's attention to here is the theme of "naturality" that pervades the book. That is, everything commutes with pull-backs–this cryptic statement will become clear upon reading the book—and this enables us to do all our calculations on subsets of $\mathbb{R}^n$, which is the only place we really know how to do calculus.

This book is an outgrowth of the author's earlier book *Differential Forms: A Complement to Vector Calculus*. In that book we introduced differential forms at a lower level, that of third semester calculus. The point there was to show how the theory of differential forms unified and clarified the material in multivariable calculus: the gradient of a function, and the curl and divergence of a vector field (in $\mathbb{R}^3$) are all "really" special cases of the exterior derivative of a differential form, and the classical theorems of Green, Stokes, and Gauss are all "really" special cases of the GST. By "really" we mean that we must first recast these results in terms of differential forms, and this is done by what we call the "Fundamental Correspondence."

However, in the (many) years since that book appeared, we have received a steady stream of emails from students and teachers who used this book, but almost invariably at a higher level. We have thus decided to rewrite it at a higher level, in order to address the needs of the actual readers of the book. Our previous book had minimal prerequisites, but for this book the reader will have to be familiar with the basics of point-set topology, and to have had a good undergraduate course in linear algebra. We use additional linear algebra material, often not covered in such a course, and we develop it when we need it.

We would like to take this opportunity to correct two historical errors we made in our earlier book. One of the motivations for developing vector calculus was, as we wrote, Maxwell's equations in electromagnetism. We wrote that Maxwell would have recognized vector calculus. In fact, the (common) expression of those equations in vector calculus terms was not due to him, but rather to Heaviside.

But it is indeed the case that this is a nineteenth century formulation, and there is an illuminating reformulation of Maxwell's equations in terms of differential forms (which we urge the interested reader to investigate). Also, Poincaré's work in celestial mechanics was another important precursor of the theory of differential forms, and in particular he proved a result now known as Poincaré's Lemma. However, there is considerable disagreement among modern authors as to what this lemma is (some say it is a given statement, others its converse). In our earlier book we wrote that the statement in one direction was Poincaré's Lemma, but we believe we got it backwards then (and correct now). See Remark 1.4.2.

We conclude with some remarks about notation and language. Results in this book have three-level numbering, so that, for example, Theorem 1.2.7 is the 7$^{\text{th}}$ numbered item in Chapter 1, Section 2. The ends of proofs are marked by the symbol □. The statements of theorems, corollaries, etc., are in italics, so are clearly delineated. But the statements of definitions, remarks, etc., are in ordinary type, so there is nothing to delineate them. We thus mark their ends by the symbol ◇. We use $A \subseteq B$ to mean that A is a subset of B, and $A \subset B$ to mean that A is a proper subset of B. We use the term "manifold" to mean precisely that, i.e., a manifold without boundary. The term "manifold with boundary" is a generalization of the term "manifold," i.e., it includes the case when the boundary is empty, in which case it is simply a manifold.

Steven H. Weintraub
Bethlehem, PA, USA
May, 2013

Contents

1 Differential Forms in $\mathbb{R}^n$, I

In this chapter we introduce differential forms in $\mathbb{R}^n$ as formal objects and show how to do algebra with them. We then introduce the operation of exterior differentiation and discuss closed and exact forms. Our discussion here is completely general, but for the sake of clarity and simplicity, we will be drawing most of our examples from $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$.

The special case $n = 3$ has some particularly interesting features mathematically, and corresponds to the world we live in physically. In Section 1.3 of this chapter, we show the correspondence between differential forms and functions/vector fields in $\mathbb{R}^3$.

1.0 Euclidean spaces, tangent spaces, and tangent vector fields

We begin by establishing some notation and by making some subtle but important distinctions that are often ignored.

We let $\mathbb{R}$ denote the set of real numbers.

DEFINITION 1.0.1. For a positive integer n, $\mathbb{R}^n$ is

$$\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}\}. \qquad \diamond$$

In dealing with $\mathbb{R}^1$, $\mathbb{R}^2$, and $\mathbb{R}^3$, we will often use (x), (x, y), and (x, y, z) rather than (x_1), (x_1, x_2), and (x_1, x_2, x_3), respectively, as coordinates.

We are about to introduce tangent spaces. For some purposes, it is convenient to have a single vector space that serves as a "model" for each tangent space, and we introduce that first.

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DEFINITION 1.0.2. For a positive integer n, $\mathbf{R}^n$ is the vector space

$$\left\{ v = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mid a_i \in \mathbb{R} \right\}.$$

with the operations

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \text{and} \quad c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}. \qquad \diamond$$

DEFINITION 1.0.3. Let $p = (x_1, \ldots, x_n) \in \mathbb{R}^n$. The tangent space $\mathbf{R}^n_p = T_p\mathbb{R}^n$ to $\mathbb{R}^n$ at p is

$$\left\{ \mathbf{v}_p = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_p \mid a_i \in \mathbb{R} \right\}. \qquad \diamond$$

Let us carefully discuss the distinction here. Elements of $\mathbb{R}^n$ are points, while for each fixed point p of $\mathbb{R}^n$, elements $\mathbf{v}_p$ of $T_p\mathbb{R}^n$ are vectors. For a fixed point p of $\mathbb{R}^n$, if $\mathbf{v}_p \in T_p\mathbb{R}^n$, we say that $\mathbf{v}_p$ is a tangent vector based at that point, so that $T_p\mathbb{R}^n$ is the vector space of tangent vectors to $\mathbb{R}^n$ at p. This is indeed a vector space as we have the operations of vector addition and scalar multiplication given by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_p + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}_p = \begin{bmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{bmatrix}_p \quad \text{and} \quad c \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_p = \begin{bmatrix} ca_1 \\ \vdots \\ ca_n \end{bmatrix}_p.$$

We say that $\mathbf{R}^n$ serves as a model for every tangent space $\mathbf{R}^n_p$ as we have an isomorphism from $\mathbf{R}^n$ to $\mathbf{R}^n_p$ given by

$$\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \mapsto \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_p.$$

Geometrically, it makes sense to add tangent vectors based at the same point of $\mathbb{R}^n$, and that is what we have done. But it does not make geometric sense to add tangent vectors based at different points of $\mathbb{R}^n$, nor does it make geometric sense to add points of $\mathbb{R}^n$, so we do not attempt to define such operations.

Let us emphasize the distinctions we have made. Often one sees points and vectors identified, but they are really different kinds of objects. Also, often one sees tangent vectors at different points identified but they are really different objects.

What we have just said is mathematically precise, but also makes sense physically. Consider, for example, a body under the gravitational influence of a star. The body has some position, i.e., it is located at some point p of $\mathbb{R}^3$, and the gravity of the star exerts a force on the body, this force being given by a vector $\mathbf{v}_p$ whose direction is toward the star and whose magnitude is given by Newton's law of universal gravitation. Furthermore $\mathbf{v}_p$ is indeed based at p as this is where the body is located, i.e., this is the point at which the force is acting. Thus we see a clear distinction between the position (point) and force (vector based at that point). If our body, located at the point p, is under the influence of a binary star system, with $\mathbf{v}_p$ the force vector for the gravitational attraction of the first star and $\mathbf{w}_p$ the force vector for the gravitational attraction of the second star, then their sum $\mathbf{v}_p + \mathbf{w}_p$ (given by the "parallelogram law") is the net gravitational force on the body, and this indeed makes sense. On the other hand, if $\mathbf{v}_p$ is a tangent vector based at p and $\mathbf{w}_q$ is a tangent vector based at some different point q, their sum $\mathbf{v}_p + \mathbf{w}_q$ is not defined, and would not make physical sense either, as what could this represent? (A body can't be in two different places at the same time!)

Next we come to the closely related notion of a tangent vector field.

DEFINITION 1.0.4. A *tangent vector field* $\mathbf{v}$ on $\mathbb{R}^n$ is a function $\mathbf{v}$ that associates to every point $p \in \mathbb{R}^n$ a tangent vector to $\mathbb{R}^n$ based at p, i.e., a tangent vector $\mathbf{v}_p \in T_p\mathbb{R}^n$. Thus we may write $\mathbf{v}(p) = \mathbf{v}_p$ for every $p \in \mathbb{R}^n$. ◇

EXAMPLE 1.0.5. For any n-tuple of real numbers $(a_1, \ldots, a_n)$, we have the *constant vector field*

$$\mathbf{v} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

given by

$$\mathbf{v}(p) = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}_p$$

for any point $p = (x_1, \dots, x_n) \in \mathbb{R}^n$.

In particular, for any i between 1 and n, if $(a_1, \dots, a_n) = (0, \dots, 0, 1, 0, \dots, 0)$ with the entry of 1 in the ith position, we denote the corresponding constant tangent vector field by $\mathbf{e}^i$ and its value at the point p by $\mathbf{e}^i(p) = \mathbf{e}^i_p$. ◇

The notation $\mathbf{e}^i$ is a bit ambiguous, as $\mathbf{e}^i$ is a vector field on $\mathbb{R}^n$ but the notation does not make clear what the value of n is. However, this will always be clear from the context.

In dealing with $\mathbb{R}^n$ for $n \le 3$, we will often use $\mathbf{i}, \mathbf{j}, \mathbf{k}$ rather than $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$.

Now, while it is improper to identify tangent vectors based at different points, it certainly is proper to consider constant tangent vector fields. (When people do identify tangent vectors based at different points, what they really should be doing is considering tangent vector fields.)

We observe that it makes sense to add two tangent vector fields: $(\mathbf{v} + \mathbf{w})_p = \mathbf{v}_p + \mathbf{w}_p$ for every point p; and to multiply a tangent vector field by a scalar: $(c\mathbf{v})_p = c\mathbf{v}_p$ for every point p. In particular, if $\mathbf{v}$ is the constant vector field in the above definition, then $\mathbf{v} = a_1\mathbf{e}^1 + a_2\mathbf{e}^2 + \cdots + a_n\mathbf{e}^n$.

As a matter of notation, we will maintain the distinction that $\mathbf{v}$ (no subscript) denotes a vector field and $\mathbf{v}_p$ denotes a tangent vector at the point p. However, often for emphasis we will use upper-case boldface letters to denote vector fields. Also, we remark that we use superscripts rather than subscripts (i.e., $\mathbf{e}^i$ rather than $\mathbf{e}_i$), as if we were to use subscripts, we would wind up using double subscripts (e.g., $\mathbf{e}_{i_p}$) and we wish to avoid that.

Now consider the physical situation of a particle moving in $\mathbb{R}^n$, say the earth orbiting the sun. If the particle is located at the point p, then we can consider its velocity vector as a tangent vector to $\mathbb{R}^n$ at p. Conversely, given a vector field $\mathbf{v}$, we may imagine that a point p represents the position of a particle and the tangent vector $\mathbf{v}_p$

represents its velocity. In this physical situation, given any point p, we may imagine that we start off at p at time $t = 0$, and move with velocity $\mathbf{v}$ for $0 \le t \le 1$, finishing up at position q at time $t = 1$. (For example, in the case of the earth orbiting the sun, with time measured in years, we would have q just about equal to p.) We can always find q by "integrating the vector field." This is something we do not want to discuss here, except to note that there is one very simple but very important case, that of a constant vector field. In this case, if p is the point $(x_1, \ldots, x_n)$ and $\mathbf{v}$ is the constant vector field $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, then q is the point

$$q = (x_1 + a_1, \ldots, x_n + a_n).$$

In this case we will write

$$q = p + \mathbf{v}.$$

(We emphasize that this notation is really an abuse of language. We are really not adding $\mathbf{v}$ to p; rather we are starting at p and following $\mathbf{v}$. But it is so concise and convenient that its virtues outweigh its vices.)

1.1 The algebra of differential forms

In this section we define differential forms and see how to do algebra with them. For the first part of our development, in the next few sections, we will be treating them as formal objects. Later on, we will of course see their true meaning.

DEFINITION 1.1.1. Let $\mathcal{R}$ be an open set in $\mathbb{R}^n$. Then

$$C^\infty(\mathcal{R}) = \{f \colon \mathcal{R} \longrightarrow \mathbb{R} \mid f \text{ has all partial derivatives of all orders at every point } p \text{ of } \mathcal{R}\}.$$

A function $f \in C^\infty(\mathcal{R})$ is said to be *smooth* on $\mathcal{R}$. ◇

For example, $f(x, y) = e^x(\sin(x + y^2))$ is a smooth function on $\mathbb{R}^2$, and $f(x, y) = 1/(x^2 + y^2)$ is a smooth function on $\mathbb{R}^2 - \{(0, 0)\}$.

DEFINITION 1.1.2. Let $\mathcal{R}$ be an open set in $\mathbb{R}^n$. Let k be a fixed nonnegative integer. A monomial k-form on $\mathcal{R}$ is an expression

$$f\,dx_{i_1}\cdots dx_{i_k},$$

where f is a smooth function on $\mathcal{R}$.

A k-form on $\mathcal{R}$ is a sum of monomial k-forms on $\mathcal{R}$.

A *differential form* φ on $\mathcal{R}$ is a k-form on $\mathcal{R}$ for some k. In this situation, k is the *degree* of φ.

We let $\Omega^k(\mathcal{R}) = \{k\text{-forms on } \mathcal{R}\}$ and $\Omega^*(\mathcal{R}) = \{\text{differential forms on } \mathcal{R}\}$. ◇

We will be using lower-case Greek letters to denote differential forms.

When dealing with differential forms in $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$, we will often use dx, dy, and dz instead of dx_1, dx_2, and dx_3, respectively.

So far, $dx_1, \ldots, dx_n$ are just symbols. (That is what we mean by saying that these are formal objects.)

We let I be a *multi-index*, i.e., $I = (i_1, \ldots, i_k)$ is a sequence of positive integers. (We allow k to be 0.) We will adopt the notation dx_I to denote the string (possibly empty) $dx_I = dx_{i_1}\cdots dx_{i_k}$. Thus a general k-form φ can be written as $\varphi = A_1 dx_{I_1} + A_2 dx_{I_2} + \cdots + A_m dx_{I_m}$, where all of $dx_{I_1}, \ldots, dx_{I_m}$ have length k. In this case, we will also say that $A_1, A_2, \ldots, A_m$ are the functions involved in φ.

In case $\varphi = A_1 dx_{I_1} + A_2 dx_{I_2} + \cdots + A_m dx_{I_m}$ with each of the functions $A_1, \ldots, A_m$ constants, we will say that φ is a constant form.

EXAMPLE 1.1.3.

(0) $\varphi = x^2 y + e^z$ is a 0-form.

(1) $\varphi = x^2\,dx + (yz+1)dz$ is a 1-form.

(2) $\varphi = xyz\,dy\,dz + xe^y\,dz\,dx + 2\,dx\,dy$ is a 2-form.

(3) $\varphi = (x^2 + xyz + z^3)dx\,dy\,dz$ is a 3-form. ◇

Addition of differential forms is done term by term. The sum $\rho = \varphi + \psi$ is only defined when φ and ψ both have the same degree k, for some k, in which case ρ also has degree k. Also, addition is required to be commutative and associative.