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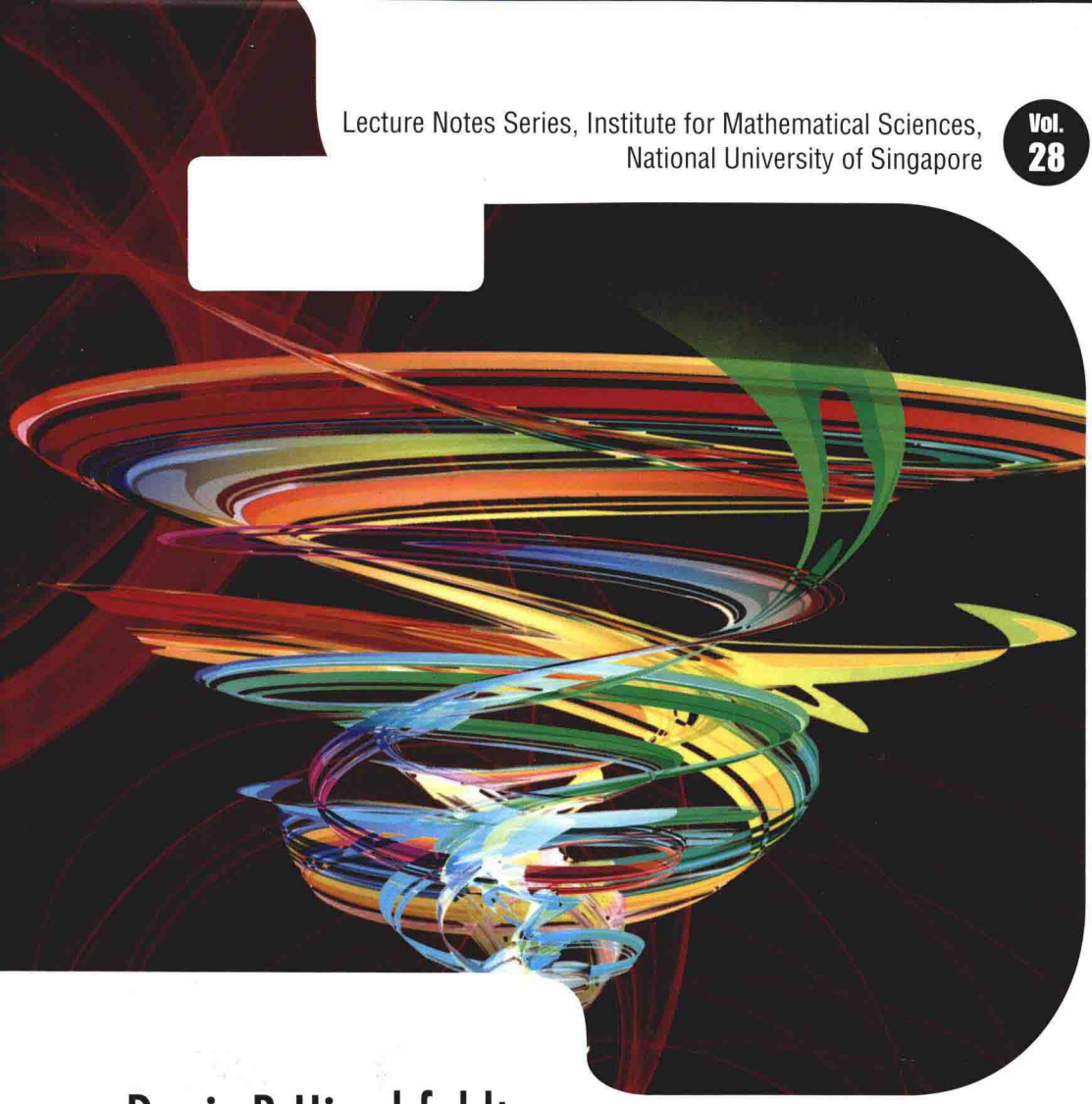

Denis R Hirschfeldt

SLICING THE TRUTH

On the Computable and Reverse Mathematics of Combinatorial Principles

Editors: Chitat Chong • Qi Feng • Theodore A Slaman • W Hugh Woodin • Yue Yang

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Foreword
by Series Editors

The Institute for Mathematical Sciences (IMS) at the National University of Singapore was established on 1 July 2000. Its mission is to foster mathematical research, both fundamental and multidisciplinary, particularly research that links mathematics to other efforts of human endeavor, and to nurture the growth of mathematical talent and expertise in research scientists, as well as to serve as a platform for research interaction between scientists in Singapore and the international scientific community.

The Institute organizes thematic programs of longer duration and mathematical activities including workshops and public lectures. The program or workshop themes are selected from among areas at the forefront of current research in the mathematical sciences and their applications.

Each volume of the *IMS Lecture Notes Series* is a compendium of papers based on lectures or tutorials delivered at a program/workshop. It brings to the international research community original results or expository articles on a subject of current interest. These volumes also serve as a record of activities that took place at the IMS.

We hope that through the regular publication of these *Lecture Notes* the Institute will achieve, in part, its objective of reaching out to the community of scholars in the promotion of research in the mathematical sciences.

July 2014

Chitat Chong
Wing Keung To
Series Editors

Foreword by Volume Editors

The series of Asian Initiative for Infinity (AII) Graduate Logic Summer School was held annually from 2010 to 2012. The lecturers were Moti Gitik, Denis Hirschfeldt and Menachem Magidor in 2010, Richard Shore, Theodore A. Slaman, John Steel, and W. Hugh Woodin in 2011, and Ilijas Farah, Ronald Jensen, Gerald E. Sacks and Stevo Todorcevic in 2012. In all, more than 150 graduate students from Asia, Europe and North America attended the summer schools. In addition, two postdoctoral fellows were appointed during each of the three summer schools. These volumes of lecture notes serve as a record of the AII activities that took place during this period.

The AII summer schools was funded by a grant from the John Templeton Foundation and partially supported by the National University of Singapore. Their generosity is gratefully acknowledged.

July 2014

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Volume Editors

Preface

When Rod Downey and I finished our book *Algorithmic Randomness and Complexity*, which was almost a decade in the making, I promised myself I would never again write a book. But accidents happen. In 2010, I was invited to give a short course at the Asian Initiative for Infinity Graduate Summer School, organized by the Institute for Mathematical Sciences and the Department of Mathematics of the National University of Singapore, and to write a version of my lecture notes for publication. The topic of the course was the reverse mathematics and computability theory of combinatorial principles, an area of research whose roots reach back several decades, but which has seen a particular surge of activity in the last few years. Much of this work has proceeded along lines that are fairly distinct from the material covered in Simpson's excellent *Subsystems of Second Order Arithmetic*, and there has been little alternative to reading research articles for those interested in understanding it. While reading original papers is highly recommended, it can be a difficult process without appropriate guidance. I wanted my notes to be an entryway into this area, providing both an overview of some fundamental ideas and techniques, and enough context to make it possible for students with at least a basic knowledge of computability theory and proof theory to appreciate the exciting advances currently happening in the area, and perhaps make contributions of their own.

I decided to adopt a case-study approach, using the study of versions of Ramsey's Theorem (for colorings of tuples of natural numbers) and related principles as illustrations of various aspects of computability theoretic and reverse mathematical analysis. Even within this deliberately narrow focus, I felt no need to be encyclopedic. It was not my goal to write a survey, but to tell a story. Nevertheless, when it comes to mathematics, a properly

illustrative story needs details. Furthermore, thorough discussions of some of these details were difficult to find in existing sources. And while there may be some who can tell a long story without digressions, I am not among them. Thus the text grew and grew, until it became a book. So be it.

I will give an overview of the book in Chapter 1, but for now, here is the abstract I wrote when still thinking of this text as an article: We discuss two closely related approaches to studying the relative strength of mathematical principles, computable mathematics and reverse mathematics. Drawing our examples from combinatorics and model theory, we explore a variety of phenomena and techniques in these areas. We begin with variations on König's Lemma, and give an introduction to reverse mathematics and related parts of computability theory. We then focus on Ramsey's Theorem as a case study in the computability theoretic and reverse mathematical analysis of combinatorial principles. We study Ramsey's Theorem for Pairs (RT^2_2) in detail, focusing on fundamental tools such as stability, cohesiveness, and Mathias forcing; and on combinatorial and model theoretic consequences of RT^2_2. We also discuss the important theme of conservativity results. In the final section, we explore several topics that reveal various aspects of computable mathematics and reverse mathematics. An appendix contains a proof of Liu's recent result that RT^2_2 does not imply Weak König's Lemma. There are exercises and open questions throughout.

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