Essential Concepts of **Differential Equations**

Volume I

Calanthia Wright

Essential Concepts of Differential Equations

Volume I

Edited by **Calanthia Wright**

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Preface

A differential equation is an integral part of the vast field of mathematics. It can be defined as a mathematical equation that relates some function of one or more variables with its derivatives. The mathematical theory of differential equations can be said to have developed together with the sciences where the equations had derived from and where the results found application or were needed. Differential equations arise whenever a deterministic relation concerning some constantly changing quantities and their rates of change in space and time is known or hypothesized. Such relations are extremely familiar and therefore differential equations play a fundamental role in many disciplines like physics, engineering, biology and economics. The mathematical theory behind the equations can also be viewed as a uniting principle behind various phenomena. The theory of conduction of heat is one of the examples of a phenomena governed by a differential equation, that is, the heat equation. One will find that there are many processes that, while seemingly different, are described by differential equations. Diverse problems, sometimes stemming from quite distinct scientific fields, may give rise to identical differential equations. Many fundamental laws of physics and chemistry can be formulated as differential equations. Even in fields such as biology and economics, differential equations can be used to represent the behavior of complex systems. Thus the arena of differential equations can be said to be quite a prolific one.

This book is an attempt to compile and collate all available research on the subject of differential equations under one umbrella. I am grateful to those who put their hard work, effort and expertise into these researches as well as those who were supportive in this endeavour. I also wish to thank my publisher for giving me this unmatched opportunity. I am extremely thankful to all the contributing authors who took out their precious time to interact with me and helped me understand their research perspectives in a better manner for the best output. Lastly, I wish to thank my family for their constant support.

Editor

Contents

On Uniform Exponential Stability and Exact Admissibility of Discrete Semigroups

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We prove that a discrete semigroup $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ of bounded linear operators acting on a complex Banach space X is uniformly exponentially stable if and only if, for each $x \in AP_0(\mathbb{Z}_+, X)$, the sequence $n \mapsto \sum_{k=0}^{n} T(n-k)x(k) : \mathbb{Z}_+ \rightarrow X$ belongs to $AP_0(\mathbb{Z}_+, X)$. Similar results for periodic discrete evolution families are also stated.

1. Introduction

The solutions of the autonomous discrete systems $x_{n+1} = Ax_n$ or $y_{n+1} = Ay_n + h_n$ lead to the idea of discrete semigroups. There are a lot of spectral criteria which characterize different types of stability (or other types of asymptotic behavior) of the solutions of above systems. For further results on asymptotic behavior of semigroups, we refer to [1].

New difficulties appear in the study of the nonautonomous systems, especially because the part of the solution generated by the forced term (h_n), that is, $\sum_{k=\nu}^{n} U(n,k)h_k$, is not a convolution in the classical sense. These difficulties may be passed by using the so-called evolution semigroups.

The evolution semigroups were exhaustively studied in [2]. Having in mind the well-known results stated in the continuous case, see for example [2, 3], we can say that this method is a very efficient one. See also [4, 5] for recent developments concerning the semigroups of evolution acting on almost periodic function spaces.

Recently, the discrete version of [6] was obtained in [7].

In this note, we study the asymptotic behavior of the discrete semigroups in terms of exact admissibility of the space of almost periodic sequences.

In this regard, we develop the theory of discrete evolution semigroups on a special space of bounded sequences. Results of this type in the continuous case may be found in [8] and the references therein. However, by contrast with the continuous case, we did not find in the existent literature papers written in the spirit of the present one referring to the discrete evolution semigroups. These results could be new and useful for people whose area of research is restricted to difference equations.

2. Definitions and Preliminary Results

Let X be a complex Banach space and $\mathscr{B}(X)$ the Banach algebra of all linear and bounded operators acting on X. The norms in X and in $\mathscr{B}(X)$ will be denoted by $\|\cdot\|$. Let $\mathbb{Z}_+$ be the set of all nonnegative integer numbers. A sequence $x : \mathbb{Z}_+ \rightarrow X$ is said to be almost periodic if for any $\epsilon > 0$ there exists an integer $l_\epsilon > 0$ such that any discrete interval of length l_ϵ contains an integer τ, such that

$$\left\|x_{n+\tau} - x_n\right\| \leq \epsilon, \quad \forall n \in \mathbb{Z}_+. \tag{1}$$

The integer number τ is called ϵ-translation number of (x_n). The set of all almost periodic sequences will be denoted by $AP(\mathbb{Z}_+, X)$. For further details about almost periodic functions, we refer to the books [9, 10]. The set $l^\infty(\mathbb{Z}_+, X)$ of all bounded sequences becomes a Banach space when it is endowed with the "sup" norm denoted by $\|\cdot\|_\infty$. Clearly, $AP(\mathbb{Z}_+, X)$ is a subset of $l^\infty(\mathbb{Z}_+, X)$. Let $P_q^0(\mathbb{Z}_+, X)$ be the space of all q-periodic ($q \geq 2$ is an integer number) sequences

x with $x(0) = 0$. Denote by $\mathcal{A}_0(\mathbb{Z}_+, X)$ the set of all sequences $\{x(n)\}_{n\geq 0}$ for which there exists $n_x \in \mathbb{Z}_+$ with $n_x > 0$ and $y_x \in P_q^0(\mathbb{Z}_+, X)$ such that

$$x(n) = \begin{cases} 0, & \forall 0 \leq n < n_x, \\ y_x(n), & \text{if } n \geq n_x. \end{cases} \quad (2)$$

Let $AP_0(\mathbb{Z}_+, X) := \overline{\text{span}}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$. Here the closeness is considered in the space $l^\infty(\mathbb{Z}_+, X)$.

For a bounded linear operator L, acting on X, we denote by $\sigma(L)$ the spectrum of L and by $\rho(L)$ its resolvent set. Recall that a subset $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ of $\mathcal{B}(X)$ is called discrete semigroup if it satisfies the following conditions:

(i) $T(0) = I$, where I is the identity operator on X.

(ii) $T(n+m) = T(n)T(m)$, for all $n, m \in \mathbb{Z}_+$.

A discrete semigroup $\mathbb{T}$ is said to be uniformly exponentially stable if there exist $N, \nu > 0$ such that

$$\|T(n)\| \leq Ne^{-\nu n} \quad \forall n \in \mathbb{Z}_+. \quad (3)$$

The spectral radius of $T(1)$ denoted by $r(T(1))$ is defined as

$$r(T(1)) := \sup\{|\lambda| : \lambda \in \sigma(T(1))\}. \quad (4)$$

It is well known that, see for example [11, page 42],

$$r(T(1)) = \lim_{n\to\infty} \|(T(1))^n\|^{1/n}. \quad (5)$$

As a consequence of (5), a discrete semigroup $\{T(n)\}_{n\in\mathbb{Z}_+}$ is uniformly exponentially stable if and only if $r(T(1)) < 1$.

Having in mind the continuous case, the "infinitesimal generator" of the discrete semigroup denoted by G is defined by $G := T(1) - I$. For discrete semigroups, the Taylor formula of order one is

$$T(n)x - x = \sum_{k=0}^{n-1} T(k)Gx, \quad \forall n \in \mathbb{Z}_+, \ n \geq 1, \ \forall x \in X. \quad (6)$$

A discrete semigroup $\mathbb{T}$ is said to be $AP_0(\mathbb{Z}_+, X)$ exact admissible, if for every $x \in AP_0(\mathbb{Z}_+, X)$ the sequence $(\sum_{k=0}^n T(n-k)h(k))_{n\in\mathbb{Z}_+}$ belongs with $AP_0(\mathbb{Z}_+, X)$.

The evolution semigroup $\mathbb{S} = \{S(n), n \in \mathbb{Z}_+\}$ associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$ is defined by

$$(S(r)x)(n) = \begin{cases} T(r)x(n-r), & \forall n \geq r, \\ 0, & 0 \leq n \leq r. \end{cases} \quad (7)$$

3. Results

The following lemma shows that the associated evolution semigroup $\{S(n)\}_{n\in\mathbb{Z}_+}$ acts on $AP_0(\mathbb{Z}_+, X)$.

Lemma 1. *Let $x \in AP_0(\mathbb{Z}_+, X)$ and $\mathbb{T} = \{T(j)\}_{j\in\mathbb{Z}_+}$ be a discrete semigroup of bounded linear operators on X. The sequence $S(r)x$, given by*

$$(S(r)x)(n) = \begin{cases} T(r)x(n-r), & \forall n \geq r \\ 0, & 0 \leq n \leq r, \end{cases} \quad (8)$$

belongs to $AP_0(\mathbb{Z}_+, X)$.

Proof. First we show that $S(r)x \in \mathcal{A}_0(\mathbb{Z}_+, X)$ for any $x \in \mathcal{A}_0(\mathbb{Z}_+, X)$. Since $x \in \mathcal{A}_0(\mathbb{Z}_+, X)$ there exist $n_x \in \mathbb{Z}_+$ with $n_x > 0$, and $(y_x(n)) \in P_q^0(\mathbb{Z}_+, X)$, such that

$$x(n) = \begin{cases} 0, & \text{if } 0 \leq n < n_x \\ y_x(n), & \text{if } n \geq n_x. \end{cases} \quad (9)$$

Let $n_{S(r)x} := r + n_x$ and set $y_{S(r)x}(\cdot) = T(r)y_x(\cdot - r)$. Clearly $y_{S(r)x}$ is q-periodic sequence. It remains to show that

$$(S(r)x)(n) = \begin{cases} 0, & \text{if } 0 \leq n < n_{S(r)x} \\ y_{S(r)x}(n), & \text{if } n \geq n_{S(r)x}. \end{cases} \quad (10)$$

If $n \leq n_{S(r)x} = r + n_x$, then $n - r < n_x$ and $x(n-r) = 0$, so

$$(S(r)x)(n) = T(r)x(n-r) = 0. \quad (11)$$

If $n \geq n_{S(r)x} = r + n_x$, then $n - r \geq n_x$ and $x(n-r) = y_x(n-r)$; hence

$$\begin{aligned}(S(r)x)(n) &= T(r)x(n-r) \\ &= T(r)y_x(n-r) \\ &= y_{S(r)x}(n).\end{aligned} \quad (12)$$

Thus $S(r)x \in \mathcal{A}_0(\mathbb{Z}_+, X)$. Now, from linearity it follows that $S(r)z$ belongs to $\text{span}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$ whenever $z \in \text{span}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$. Let now $\epsilon > 0$, $x \in AP_0(\mathbb{Z}_+, X)$, and let $z \in \text{span}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$, such that $\|x - z\|_{l^\infty(\mathbb{Z}_+, X)} < \epsilon$. Clearly $S(r)z$ belongs to $\text{span}\{\mathcal{A}_0(\mathbb{Z}_+, X)\}$, and

$$\begin{aligned}\|S(r)z - S(r)x\|_{l^\infty(\mathbb{Z}_+, X)} &= \sup_{n\geq r} \|T(r)[z(n-r) - x(n-r)]\| \\ &\leq Me^{\nu r} \sup_{n\geq r} \|z(n-r) - x(n-r)\| \\ &\leq Me^{\nu r}\epsilon,\end{aligned} \quad (13)$$

that is, $S(r)x$ is in $AP_0(\mathbb{Z}_+, X)$. This completes the proof.

Lemma 2. *Let $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup of bounded linear operators on X, and let $\mathbb{S} = \{S(n), n \in \mathbb{Z}_+\}$ be the evolution semigroup associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$, having $G_\mathbb{S}$ as generator. Let $x, z \in AP_0(\mathbb{Z}_+, X)$. The following two statements are equivalent:*

(i) $G_\mathbb{S}x = -z$,

(ii) $x(n) = \sum_{k=0}^n T(n-k)z(k)$, for all $n \in \mathbb{Z}_+$.

Proof. (i) ⇒ (ii): Using the Taylor formula (6), one has

$$S(n)x - x = \sum_{m=0}^{n-1} S(m)G_\mathbb{S}x = -\sum_{m=0}^{n-1} S(m)z. \quad (14)$$

Then, for every $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} x(n) &= (S(n)x)(n) + \sum_{m=0}^{n-1}(S(m)z)(n) \\ &= T(n)x(0) + \sum_{m=0}^{n-1}T(m)z(n-m) \\ &= \sum_{k=0}^{n}T(n-k)z(k). \end{aligned} \tag{15}$$

(ii) $\Rightarrow$ (i): For each $n \in \mathbb{Z}_+$, one has

$$\begin{aligned} (G_{\mathbb{S}}x)(n) &= (S(1)-I)x(n) \\ &= T(1)x(n-1) - x(n) \\ &= T(1)\sum_{k=0}^{n-1}T(n-1-k)z(k) - x(n) \\ &= \sum_{k=0}^{n-1}T(n-k)z(k) - \sum_{k=0}^{n}T(n-k)z(k) \\ &= -z(n). \end{aligned} \tag{16}$$

This completes the proof.

See also [12], for a variant of this lemma in other space.

The next result is the main ingredient in the proof of Theorem 5 that follows.

Theorem 3 (see [7]). *Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X, and let μ be a real number. If*

$$\sup_{n\geq 0}\left\|\sum_{k=0}^{n}e^{i\mu k}T(n-k)f(k)\right\| < \infty, \tag{17}$$

for every $f \in P_0^q(\mathbb{Z}_+, X)$, then $T(1)$ is power bounded (i.e., $\sup_{n\in\mathbb{Z}_+}\|A^n\| < \infty$) and $e^{i\mu} \in \rho(T(1))$.

As a corollary of this theorem, we state the following.

Corollary 4 (see [7]). *Let $\mathbb{T} = \{T(n) : n \in \mathbb{Z}_+\}$ be a discrete semigroup on X. If the condition (17) holds for every $\mu \in \mathbb{R}$ and every f in $P_0^q(\mathbb{Z}_+, X)$, then the semigroup $\mathbb{T}$ is uniformly exponentially stable.*

The result of this paper reads as follows.

Theorem 5. *Let $\mathbb{T} = \{T(n)\}_{n\in\mathbb{Z}_+}$ be a discrete semigroup on X. The following four statements are equivalent:*

(i) *$\mathbb{T}$ is uniformly exponentially stable.*

(ii) *The evolution semigroup $\mathbb{S}$ associated with $\mathbb{T}$ on $AP_0(\mathbb{Z}_+, X)$ is uniformly exponentially stable.*

(iii) *The semigroup $\mathbb{T}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.*

(iv) *$\sup_{n\in\mathbb{Z}_+}\|\sum_{k=0}^{n}T(n-k)z(k)\| = M_z < \infty$, for all $z \in AP_0(\mathbb{Z}_+, X)$.*

Proof. (i) $\Rightarrow$ (ii): Let $\mathbb{T}$ be uniformly exponentially stable, and let N and ν be positive constants such that

$$\|T(n)\| \leq Ne^{-\nu n} \quad \forall n \in \mathbb{Z}_+. \tag{18}$$

Then for every f in $AP_0(\mathbb{Z}_+, X)$, one has

$$\|S(j)f\|_\infty = \sup_{n\geq j}\|T(j)f(n-j)\| \leq Ne^{-\nu j}\|f\|_\infty. \tag{19}$$

(ii) $\Rightarrow$ (iii): Since $\mathbb{S}$ is uniformly exponentially stable, $1 \in \rho(S(1))$, that is, $S(1) - I$ is invertible. Then for each z in $AP_0(\mathbb{Z}_+, X)$, there exists $u \in AP_0(\mathbb{Z}_+, X)$ such that $(S(1) - I)u = -z$.

On the other hand, by Lemma 2, $u(n) = \sum_{k=0}^{n}T(k)z(n-k)$, for every $n \in \mathbb{Z}_+$; hence $\mathbb{T}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.

(iii) $\Rightarrow$ (iv) It is obvious.

(iv) $\Rightarrow$ (i) Obviously, if $z \in P_q^0(\mathbb{Z}_+, X)$ and μ is a real number, then $(e^{i\mu n}z(n))_{n\in\mathbb{Z}_+}$ belongs to $AP_0(\mathbb{Z}_+, X)$. Now, we can apply Corollary 4 to finish the proof.

The following example is a concrete application of Theorem 5.

Example 6. Let X be a complex Banach space, and let A be a bounded linear operator acting on X. Consider the following two discrete Cauchy problems:

$$\begin{aligned} x_{j+1} &= Ax_j, \quad j \in \mathbb{Z}_+, \\ x_0 &= b, \end{aligned} \tag{20}$$

$$\begin{aligned} y_{j+1} &= Ay_j + f_{j+1}, \quad j \in \mathbb{Z}_+, \\ y_0 &= 0. \end{aligned} \tag{21}$$

The solutions of (20) and (21) are (resp.) given by $x_j = T(j)b$ and $y_j = \sum_{k=0}^{j}T(j-k)x(k)$. Here $T(k) := A^k$.

From Theorem 5, the following two statements are equivalent.

(1) For each $b \in X$ the solution of (20) decays exponentially, or, equivalently, there exist two positive constants K and ν such that

$$\|T(j)x\| \leq Ke^{-\nu j}\|x\| \quad \forall x \in X. \tag{22}$$

(2) For each $f \in AP_0(\mathbb{Z}_+, X)$ the solution of (21) belongs to $AP_0(\mathbb{Z}_+, X)$.

In fact, we can state a more general result concerning q-periodic discrete evolution families. To establish this result, we recall that a family $\mathcal{U} = \{U(n,m) : n \geq m \in \mathbb{Z}_+\} \subset \mathcal{B}(X)$ is said to be q-periodic discrete evolution family if it satisfies the following properties.

(i) $U(n,n) = I$ and $U(n,m)U(m,r) = U(n,r)$, for all $n, m, r \in \mathbb{Z}_+$ with $n \geq m \geq r \in \mathbb{Z}_+$, where I is the identity operator on X.

(ii) $U(n+q, m+q) = U(n,m)$, for all $n \geq m \in \mathbb{Z}_+$.

It is said to be uniformly exponentially stable if there exist the positive constants K and ν such that

$$\|U(n,m)\| \le Ke^{-\nu(n-m)} \quad \forall m \ge n \in \mathbb{Z}_+. \tag{23}$$

Also, the family $\mathcal{U}$ is said to be $AP_0(\mathbb{Z}_+, X)$ exact admissible, if for every $z \in AP_0(\mathbb{Z}_+, X)$ the sequence $(\sum_{k=0}^{n} U(n, k)z(k))_{n\in\mathbb{Z}_+}$ belongs to $AP_0(\mathbb{Z}_+, X)$.

The discrete evolution semigroup $\mathcal{T} = \{\mathcal{T}(n), n \in \mathbb{Z}_+\}$ associated with the evolution family $\mathcal{U}$ on $AP_0(\mathbb{Z}_+, X)$ is defined by

$$(\mathcal{T}(n)z)(r) = \begin{cases} U(r, r-n)z(r-n), & \forall r \ge n, \\ 0, & \text{otherwise.} \end{cases} \tag{24}$$

As in Lemma 1 it can be proved that it acts on $AP_0(\mathbb{Z}_+, X)$.

Theorem 7. *Let $\mathcal{U} = \{U(n,m) : n \ge m \in \mathbb{Z}_+\}$ be a q-periodic evolution family of bounded linear operators on X. The following statements are equivalent:*

(1) *$\mathcal{U}$ is uniformly exponentially stable.*

(2) *The evolution semigroup $\mathcal{T}$ associated with $\mathcal{U}$ is uniformly exponentially stable.*

(3) *$\mathcal{U}$ is $AP_0(\mathbb{Z}_+, X)$ exact admissible.*

(4) $\sup_{n\in\mathbb{Z}_+} \| \sum_{k=0}^{n} U(n,k)h(k)\| < \infty$, *for all $h \in AP_0(\mathbb{Z}_+, X)$.*

The proofs of (1) ⇒ (2) ⇒ (3) ⇒ (4) are similar to those in the semigroup case. For the proof of (4) ⇒ (1) we use the following result from [13].

If for every $\mu \in \mathbb{R}$ and every $z \in P_q^0(\mathbb{Z}_+, X)$, one has

$$\sup_{n\in\mathbb{Z}_+} \left\| \sum_{k=0}^{n} e^{i\mu k} U(n,k) z(k) \right\| := M(\mu, z) < \infty, \tag{25}$$

then the family $\mathcal{U}$ is uniformly exponentially stable.

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2

Oscillations of a Class of Forced Second-Order Differential Equations with Possible Discontinuous Coefficients

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We study the oscillation of all solutions of a general class of forced second-order differential equations, where their second derivative is not necessarily a continuous function and the coefficients of the main equation may be discontinuous. Our main results are not included in the previously published known oscillation criteria of interval type. Many examples and consequences are presented illustrating the main results.

1. Introduction

Let $t_0 > 0$ and let $AC_{\text{loc}}([t_0,\infty),\mathbb{R})$ denote the set of all real functions absolutely continuous on every bounded interval $[a,b] \subset [t_0,\infty)$. We study the oscillatory behaviour of all solutions $x = x(t)$ of the following class of forced second-order differential equations:

$$\left(r(t)\,\Phi\left(x(t),x'(t)\right)\right)' + q(t)\,f(x(t)) = e(t), \quad \text{a.e. in } [t_0,\infty), \tag{1}$$

$$x, r\Phi\left(x,x'\right) \in AC_{\text{loc}}\left([t_0,\infty),\mathbb{R}\right),$$

where the functions $\Phi : \mathbb{R}^2 \to \mathbb{R}$, $\Phi = \Phi(u,v)$, $f : \mathbb{R} \to \mathbb{R}$, and $f = f(u)$ satisfy some general conditions given in Section 2. A continuous function $x = x(t)$ is said to be *oscillatory* if there is a sequence $t_n \in [t_0,\infty)$, such that $x(t_n) = 0$ for all $n \in \mathbb{N}$ and $t_n \to \infty$ as $n \to \infty$. A differential equation is *oscillatory* if all its solutions are oscillatory.

The forcing term $e(t)$ is a sign-changing function (possibly discontinuous). This can be formulated by the following hypothesis: for every $T \geq t_0$ there exist two intervals (a_1,b_1) and (a_2,b_2), $T \leq a_1 < b_1 \leq a_2 < b_2$, such that

$$e(t) \geq 0, \quad t \in (a_1,b_1), \qquad e(t) \leq 0, \quad t \in (a_2,b_2). \tag{2}$$

The coefficient $q(t)$ may be a discontinuous function on $[t_0,\infty)$ and the case $x \notin C^2((t_0,\infty),\mathbb{R})$ occurs in our main results and examples too. Two important classes of functions $\Phi(u,v)$ are included in the differential operator $(r(t)\Phi(x,x'))'$ as

$$\Phi(u,v) = \phi(u)\,v, \quad \Phi(u,v) = \frac{\phi(u)\,v}{\sqrt{1+v^2}}, \quad (u,v) \in \mathbb{R}^2. \tag{3}$$

The first one is the classic second-order differential operator which is linear in x' and the second one is the so-called one-dimensional mean curvature differential operator; see Examples 1 and 2.

Depending on $q(t)$, we propose the following four simple models for (1):

(i) $q(t)$ is strictly positive and continuous on $[t_0,\infty)$ as

$$x'' + 4m^2 f(x) = h(\sin(mt)), \quad \text{a.e. in } [t_0,\infty), \qquad x, x' \in AC_{\text{loc}}([t_0,\infty),\mathbb{R}), \quad x \notin C^2((t_0,\infty),\mathbb{R}); \tag{4}$$

(ii) $q(t)$ is nonnegative and continuous on $[t_0,\infty)$ as

$$x'' + m^2\pi^2[\cos(mt)]^+ f(x) = h(\sin(mt)), \quad \text{a.e. in } [t_0,\infty), \tag{5}$$

$$x, x' \in AC_{\text{loc}}([t_0,\infty),\mathbb{R});$$

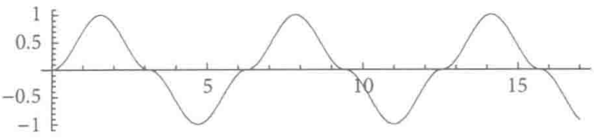

FIGURE 1: Function $x(t) = |\sin(t)|\sin(t)$ is a solution of (4).

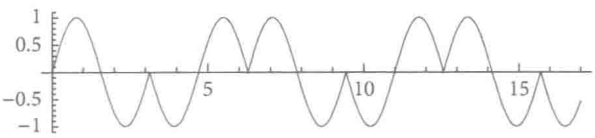

FIGURE 2: $x'(t) = 2|\sin(t)|\cos(t)$ and hence $x''(t)$ is not a continuous function.

(iii) $q(t)$ is nonnegative and discontinuous on $[t_0, \infty)$ as

$$x'' + 4m^2[\operatorname{sign}(\cos(mt))]^+ f(x) = h(\sin(mt)), \quad \text{a.e. in } [t_0, \infty), \qquad (6)$$
$$x, x' \in AC_{\text{loc}}([t_0, \infty), \mathbb{R});$$

(iv) $q(t)$ is sign changing and discontinuous on $[t_0, \infty)$ as

$$x'' + 4m^2 \operatorname{sign}(\cos(mt)) f(x) = h(\sin(mt)), \quad \text{a.e. in } [t_0, \infty), \qquad (7)$$
$$x, x' \in AC_{\text{loc}}([t_0, \infty), \mathbb{R}),$$

where $m \in \mathbb{N}$ and $h(s)$ is an arbitrary function such that $h(s)\, s > 0$ for all $s \neq 0$, for instance, $h(s) = s$ or $h(s) = \operatorname{sign}(s)$. According to Corollaries 7 and 10, we will show that (4)–(7) are oscillatory provided the function $f = f(u)$ satisfies $f(u)/u \geq K \geq 1$ for all $u \neq 0$; see Examples 8–13. It is interesting that in particular for $f(u) = u$ and $h(s) = 2m^2 \operatorname{sign}(s)$, (4) allows an explicit oscillatory solution $x(t) = |\sin(mt)|\sin(mt)$ as shown in Figures 1 and 2.

Moreover, as a consequence of Corollary 7, one can show that all solutions of (4) are oscillatory; for details see Example 8. The main goal of this paper is to give some sufficient conditions on functions $\Phi(u, v)$, $f(u)$ and the coefficients $r(t)$, $q(t)$, and $e(t)$ such that (1) is oscillatory; see Theorems 3 and 4. It will also cover the model equations (4)–(7) as well as some other examples presented in Section 2.

To the best of our knowledge, it seems that there are only few papers which study the oscillation of the second-order differential equations with nonsmooth (local integrable) coefficients; see [1–3]. More precisely, in [1] the author studied the interval oscillation criteria for the following second-order half-linear differential equation:

$$\left(r(t)\left|x'(t)\right|^{\sigma-1} x'(t)\right)' + q(t)|x(t)|^{\sigma-1} x(t) = 0, \quad \text{a.e. in } (0, \infty), \qquad (8)$$
$$x, r\left|x'\right|^{\sigma-1} x' \in AC_{\text{loc}}((0, \infty), \mathbb{R}),$$

where $\sigma > 1$ and $1/r$, $q \in L_{\text{loc}}((0, \infty), \mathbb{R})$ such that $\int_0^\infty r^{-1/\sigma}(t)dr = \infty$. See also [2] but with the solution space $C^1((0, \infty), \mathbb{R})$ instead of $AC_{\text{loc}}((0, \infty), \mathbb{R})$, that is, x and $r|x'|^{\sigma-1}x' \in C^1((0, \infty), \mathbb{R})$.

In [3], the authors consider the following second-order differential equation:

$$\left(r(t)x'(t)\right)' + Q\left(t, x(t), x'(t)\right) = 0, \quad t \geq t_0, \qquad (9)$$

where $r(t) > 0$ a.e. in $[t_0, \infty)$, $1/r \in L_{\text{loc}}([t_0, \infty), \mathbb{R})$, and $Q(t, y, z)$ is locally integrable function in t and continuous in (y, z). Equation (9) allows the forcing term $e(t)$ in the next sense as follows:

$$yQ(t, y, z) \geq q(t)\, yf(y) - e(t)\, y \quad \forall (t, y, z) \in [t_0, \infty) \times \mathbb{R}^2, \qquad (10)$$

where $e = e(t)$ satisfies (2), but the functions $q = q(t)$ and $f = f(y)$ are smooth enough in their variables, that is, $q \in C([t_0, \infty), \mathbb{R})$ and $f \in C^1(\mathbb{R}, \mathbb{R})$.

On certain oscillation criteria for various classes of forced second-order differential equations with continuous coefficients, we refer the reader to [4–13]. Our method modifies a recently used one in [14, 15] and it contains the classic Riccati transformation of the main equation, a blow-up argument and pointwise comparison principle. The comparison principle applies to all sub- and supersolutions of a class of the generalized Riccati differential equations with nonlinear terms that are supposed to be locally integrable in the first variable and locally Lipschitz continuous in the second variable.

2. Hypotheses, Results, and Consequences

First of all, the function $\Phi(u, v)$ which appears in the second-order differential operator of (1) satisfies

$$|u|^{\gamma-2} v\Phi(u, v) \geq g(|\Phi(u, v)|) \quad \forall u, v \in \mathbb{R}, \qquad (11)$$

where $\gamma \geq 2$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a locally Lipschitz function $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$g(cs) \geq c^\gamma g_0(s) \quad \forall c > 0,\ s > 0,$$
$$g_0(s) + M_0 \geq s^2 \quad \text{for some } M_0 \geq 0 \text{ and all } s \in \mathbb{R}_+. \qquad (12)$$

In most cases, $g(s) = g_0(s)g_1(s)$, $s > 0$, where $g_0(s) = s^\gamma$, $\gamma \geq 2$, and $g_1(s)$ is an arbitrary function satisfying $g_1(s) \geq 1$. Thus, for such $g(s)$ with $g_1(s) \equiv 1$, condition (11) became:

$$|u|^{\gamma-2} v\Phi(u, v) \geq |\Phi(u, v)|^\gamma \quad \forall u, v \in \mathbb{R}. \qquad (13)$$

It is not difficult to check that if $g_0 \in C^1(\mathbb{R}_+)$ or $g_0(s)$ is a convex function, then it is locally Lipschitz on $\mathbb{R}_+$ too; see for instance [16, Theorem 1.3.3].

Two essential classes of the second-order differential operators $(r(t)\Phi(x, x'))'$ satisfy condition (13), as is shown in the next examples.

Example 1. We consider the second-order differential operator which is linear in x' as follows:

$$\left(r(t)\Phi\left(x,x'\right)\right)' = \alpha\left(\phi(x)x'\right)', \tag{14}$$

where $\alpha > 0$ and $0 \le \phi(u) \le 1$ for all $u \in \mathbb{R}$. Obviously, the function $\Phi(u,v) = \phi(u)v$ satisfies condition (13) in particular for $\gamma = 2$. Two usual choices for $\phi(u)$ are $\phi(u) = |\sin u|$ and $\phi(u) = |u|/(1+|u|)$.

Example 2. We consider a quasilinear differential operator (the so-called one-dimensional prescribed mean curvature operator) as follows:

$$\left(r(t)\Phi\left(x,x'\right)\right)' = \alpha\left(\phi(x)\frac{x'}{\sqrt{1+x'^2}}\right)', \tag{15}$$

where $\alpha > 0$ and $0 \le \phi^{\gamma-1}(u) \le |u|^{\gamma-2}$ for all $u \in \mathbb{R}$. It is not difficult to check that condition (13) is satisfied in particular for $\Phi(u,v) = \phi(u)v/(1+v^2)^{1/2}$ and for any $\gamma > 1$. For $\phi(u)$, we can take the same choice as in the previous example.

Next, we suppose the existence of a constant K such that

$$\frac{f(u)}{u} \ge K > 0 \quad \forall u \ne 0. \tag{16}$$

In order to simplify our consideration here, in many examples we often use $f(u) = Ku$.

Condition (2) means that there exists a sequence of pairs of intervals $J_{1j} = [a_{1j}, b_{1j}]$ and $J_{2j} = [a_{2j}, b_{2j}]$, $j \in \mathbb{N}$, contained in (t_0, ∞), such that the sequences $(a_{1j})_{j\ge1}$, $(b_{1j})_{j\ge1}$, $(a_{2j})_{j\ge1}$, and $(b_{2j})_{j\ge1}$ are increasing, $a_{1j} < b_{1j} \le a_{2j} < b_{2j}$ for each j, and

$$e(t) \ge 0 \quad \text{on } J_{1j}, \quad e(t) \le 0 \quad \text{on } J_{2j} \quad \text{for each } j \in \mathbb{N}, \qquad \lim_{j\to\infty} a_{1j} = \infty. \tag{17}$$

On the intervals J_{1j} and J_{2j}, the coefficient $r(t)$ satisfies

$$r(t) > 0 \quad \text{on } J_{ij}, r^{1-\gamma} \in L^1\left(J_{ij}\right) \quad \forall i \in \{1,2\},\ j \in \mathbb{N}. \tag{18}$$

Let there be a real function $C = C(t)$, $C \in L^1_{\text{loc}}((t_0,\infty),\mathbb{R})$, and let there exist a sequence of positive real numbers $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$C(t) \ge 0 \quad \text{on } J_{ij}, c_{ij} := \int_{J_{ij}} C(\tau)\,d\tau > 0 \quad \forall i \in \{1,2\},\ j \in \mathbb{N},$$
$$\frac{1}{c_{ij}}C(t) \le \frac{1}{\pi}\min\left\{\left(\lambda_j r(t)\right)^{1-\gamma}, \frac{K}{M_0+1}\lambda_j q(t)\right\} \quad \forall t \in J_{ij},\ i \in \{1,2\},\ j \in \mathbb{N}, \tag{19}$$

where γ, M_0, and K are constants defined in (11), (12), and (16), respectively.

The proof of the following main result will be presented in Section 4.

Theorem 3. *Let the functions $\Phi(u,v)$, $f(u)$, $e(t)$, and $r(t)$ satisfy* (11), (12), (16), (17), *and* (18), *respectively. Let $q(t) \ge 0$ and $q(t) \not\equiv 0$ on each interval J_{ij}, $i \in \{1,2\}$, $j \in \mathbb{N}$. If* (19) *is fulfilled, then* (1) *is oscillatory.*

Condition (19) can be replaced by an equivalent one, which has a more practical value and takes a simpler form since we do not need a sequence of auxiliary parameters $(\lambda_j)_{j\in\mathbb{N}}$: let there be a real function $C = C(t)$, $C \in L^1_{\text{loc}}((t_0,\infty),\mathbb{R})$ such that

$$C(t) \ge 0 \quad \text{on } J_{ij}, c_{ij} := \int_{I_{ij}} C(\tau)\,d\tau > 0 \quad \forall i \in \{1,2\},\ j \in \mathbb{N},$$
$$\sup_{t\in J_{ij}}\left[\frac{C(t)}{q(t)}\right]\sup_{t\in J_{ij}}\left[r(t)C(t)^{1/(\gamma-1)}\right] \le \frac{K}{M_0+1}\left(\frac{c_{ij}}{\pi}\right)^{\gamma/(\gamma-1)} \quad \forall i \in \{1,2\},\ j \in \mathbb{N}, \tag{20}$$

where γ, M_0, and K are constants defined in (11), (12), and (16), respectively. Since we will show that (19) and (20) are equivalent, see page 8, the next oscillation criterion immediately follows from Theorem 3.

Theorem 4. *Let the functions $\Phi(u,v)$, $f(u)$, $e(t)$, and $r(t)$ satisfy* (11), (12), (16), (17), *and* (18), *respectively. Let $q(t) \ge 0$ and $q(t) \not\equiv 0$ on each interval J_{ij}, $i \in \{1,2\}$, $j \in \mathbb{N}$. If* (20) *is fulfilled, then* (1) *is oscillatory.*

Remark 5. Assuming that $L := \lim_{j\to\infty} a_{1j} < \infty$, we can ensure the oscillation in the point L. Note that $L = \lim_{j\to\infty} a_{2j}$ since $a_{1j} < a_{2j} < a_{1j+1}$. Thus, we can generate a one-sided (right) limit.

Now, we consider some consequences of Theorem 4, which depend on the qualitative properties of the coefficient $q(t)$.

Substituting $C(t) \equiv 1$ in (20), Theorem 4 implies the following result involving lower bounds on the lengths of intervals $|J_{ij}| = b_{ij} - a_{ij}$.

Corollary 6 ($q(t)$ is positive). *Let the functions $\Phi(u,v)$, $f(u)$, $e(t)$, and $r(t)$ satisfy* (11), (12), (16), (17), *and* (18), *respectively. Let $\inf_{t\in J_{ij}} q(t) > 0$ for each $i \in \{1,2\}$, $j \in \mathbb{N}$. If*

$$\left|J_{ij}\right| \ge \pi\left(\frac{(M_0+1)\sup_{t\in J_{ij}} r(t)}{K\inf_{t\in J_{ij}} q(t)}\right)^{(\gamma-1)/\gamma} \quad \forall i \in \{1,2\},\ j \in \mathbb{N}, \tag{21}$$

then (1) *is oscillatory.*

Corollary 7 ($q(t)$ is bounded from below by a positive constant). *Let the functions $\Phi(u,v)$, $f(u)$, $e(t)$, and $r(t)$ satisfy* (11), (12), (16), (17), *and* (18), *respectively. Let there be two constants r_0, q_0 satisfying*

$$0 < r(t) \le r_0, \quad q(t) \ge q_0 > 0 \quad \forall i \in \{1,2\}, \ j \in \mathbb{N}. \qquad (22)$$

If

$$\left|J_{ij}\right| \ge \pi \left(\frac{(M_0+1)\,r_0}{K q_0} \right)^{(\gamma-1)/\gamma} \quad \forall i \in \{1,2\}, \ j \in \mathbb{N}, \qquad (23)$$

then (1) *is oscillatory, where γ, M_0, and K are constants defined in* (11), (12), *and* (16), *respectively.*

Example 8 (oscillation of (4)). We know that $x(t) = |\sin(mt)|\sin(mt)$ is an oscillatory solution of (4). However, according to Corollary 7, we can show that all solutions of (4) are oscillatory. Indeed, since $\Phi(u,v) \equiv v$, the conditions (11) and (12) are satisfied especially for $\gamma = 2$, $g(s) = g_0(s) = s^2$ and $M_0 = 0$. Next, $f(u) \equiv u$ implies that condition (16) is satisfied especially for $K = 1$. Since $r(t) \equiv 1$ and $q(t) \equiv 4m^2$, it is clear that conditions (18) and (22) are also satisfied in particular for $r_0 = 1$ and $q_0 = 4m^2$. Moreover, since $e(t) = h(\sin(mt))$ and $h(s)s > 0$, $s \neq 0$, we have that (17) is fulfilled for $a_{1j} = 2j\pi/m$, $b_{1j} = (2j+1)\pi/m = a_{2j}$ and $b_{2j} = (2j+2)\pi/m$. Moreover,

$$\left|J_{ij}\right| = b_i - a_i = \frac{\pi}{m} > 0, \quad i \in \{1,2\}. \qquad (24)$$

Hence, we conclude that the required condition (23) is fulfilled, that is,

$$\left|J_{ij}\right| = \frac{\pi}{m} \ge \frac{\pi}{\sqrt{4m^2}} = \pi \left(\frac{(M_0+1)\,r_0}{q_0} \right)^{(\gamma-1)/\gamma} \ge \pi \left(\frac{(M_0+1)\,r_0}{K q_0} \right)^{(\gamma-1)/\gamma}. \qquad (25)$$

Thus, all conditions of Corollary 7 are satisfied and hence (4) is oscillatory.

Example 9. We consider the following class of equations:

$$x'' - 2\frac{S''(t)}{S(t)}\, x = 2\,\mathrm{sign}\,(S(t))\, S'^2(t), \quad \text{a.e. in } [t_0, \infty),$$
$$x, x' \in AC_{\mathrm{loc}}([t_0,\infty), \mathbb{R}), \quad x \notin C^2((t_0,\infty), \mathbb{R}), \qquad (26)$$

where $S = S(t)$, $S \in C^2(\mathbb{R})$ is an oscillatory function such that the zeros t_n of the function $\mathrm{sign}(S(t))S'^2(t)$ satisfy $t_n \to \infty$, there is a $\tau_0 \in \mathbb{R}$ such that $t_{n+1} - t_n \ge \tau_0 > 0$ for all $n \in \mathbb{N}$, and $S(t) \neq 0$ on (t_n, t_{n+1}). This equation allows an explicitly given oscillatory solution $x(t) = |S(t)|S(t)$. Moreover, if there is a constant $s_0 > 0$ such that

$$-2\frac{S''(t)}{S(t)} \ge s_0, \quad t \in (t_n, t_{n+1}), \quad \tau_0 \ge \frac{\pi}{\sqrt{s_0}}, \qquad (27)$$

then by Corollary 7 we conclude that (26) is oscillatory. Indeed, conditions (11), (12), and (16) are satisfied by the same reasons as in Example 8. Condition (18) is satisfied because of $r(t) \equiv 1$. Also, from (27) it follows that (22) and (23) are fulfilled in particular for $\gamma = 2$, $M_0 = 0$, $r_0 = 1$, $q_0 = s_0$, and $K \ge 1$, that is,

$$\left|J_{ij}\right| = \left|t_{j+1} - t_j\right| \ge \tau_0 \ge \frac{\pi}{\sqrt{s_0}} = \pi \left(\frac{(M_0+1)\,r_0}{q_0} \right)^{(\gamma-1)/\gamma} \ge \pi \left(\frac{(M_0+1)\,r_0}{K q_0} \right)^{(\gamma-1)/\gamma}. \qquad (28)$$

Hence Corollary 7 proves this result.

As the second consequence of Theorem 3 is unlike the first one, we consider the case when the coefficient $q(t)$ is not a strictly positive function. Here by $\{q = 0\}$ we denote the set of all $t \in \mathbb{R}$ such that $q(t) = 0$.

Corollary 10 ($q(t)$ is nonnegative, but not $\equiv 0$). *Let the functions $\Phi(u,v)$, $f(u)$, $e(t)$, and $r(t)$ satisfy* (11), (12), (16), (17), *and* (18), *respectively. Let $q(t) \ge 0$ on each interval J_{ij}, $i \in \{1,2\}$, $j \in \mathbb{N}$, such that*

$$q_{ij} := \int_{J_{ij}} q(\tau)\, d\tau > 0, \quad i \in \{1,2\}, \ j \in \mathbb{N}. \qquad (29)$$

If

$$\frac{q_{ij}^{\gamma}}{q(t)} \ge \pi^{\gamma} \left(\frac{(M_0+1)\, r(t)}{K} \right)^{\gamma-1}, \quad t \in J_{ij} \setminus \{q = 0\}, \ i \in \{1,2\}, \ j \in \mathbb{N}, \qquad (30)$$

then (1) *is oscillatory.*

Proof. It suffices to show that (30) is equivalent to the existence of a real number λ_j such that

$$\lambda_j \ge \frac{\pi(M_0+1)}{K}\frac{1}{q_{ij}}, \qquad \frac{1}{q_{ij}} q(t) \le \frac{1}{\pi}\left(\lambda_j r(t)\right)^{1-\gamma}, \quad t \in J_{ij}, \ i \in \{1,2\}, \ j \in \mathbb{N}. \qquad (31)$$

The claim will then follow from Theorem 3. Inequality (31) is for any $t \in J_{ij} \setminus \{q = 0\}$ equivalent to

$$\frac{\pi(M_0+1)}{K}\frac{1}{q_{ij}} \le \lambda_j \le \left(\frac{q_{ij}}{\pi q(t)} \right)^{1/(\gamma-1)} \frac{1}{r(t)}, \qquad (32)$$

that is, to

$$\frac{\pi(M_0+1)}{K}\frac{1}{q_{ij}} \le \left(\frac{q_{ij}}{\pi q(t)} \right)^{1/(\gamma-1)} \frac{1}{r(t)}. \qquad (33)$$

This inequality is easily seen to be equivalent to (30) for any $t \in J_{ij} \setminus \{q = 0\}$. Note that if $t \in \{q = 0\}$, then the second inequality in (31) is trivially satisfied.