New Perspectives on

Mathematical Practices

Essays in Philosophy and History of Mathematics

edited by
Bart Van Kerkhove

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EDITORIAL PREFACE

The *Second Perspectives on Mathematical Practices Conference* (PMP2007) was held at the Free University of Brussels (VUB), Belgium, from 26 to 28 March, 2007. This volume contains texts issuing from talks delivered at that occasion which particularly focused on the historical dimension of mathematical practice, the core subject of the conference. All papers gathered here address aspects of the question how the philosophy of mathematics relates to the history of mathematics. Nature and goals of this type of inquiry have been most clearly stated by José Ferreirós and Jeremy Gray in the introduction to their seminal reader *The Architecture of Modern Mathematics* (Oxford University Press, 2006), opposing with great sensitivity the ahistorical received view in the philosophy of mathematics to a recently emerging trend of studies in contextualized mathematical practices. We subscribe to the programme set out by them, and hope to provide here with a modest contribution to it. Incidentally, both editors have participated to PMP2007, and one may find Gray's paper included here, while that by Ferreirós is part of a companion volume consisting of rather philosophically laden texts, to appear with *College Publications.*

Let me give, at the outset of this collection, an overview of the articles included, which have been loosely organized in chronological order of the period covered by them. Algebra is the topic of the first triple of papers, opening with that by *Albrecht Heeffer* (Ghent), in which it is contended that the traditional three-stage division of the development of algebra, viz. into rhetorical, syncopated and symbolic phases, is not adequate. As is argued, it would better be replaced by an alternative account, in which non-symbolic, proto-symbolic and symbolic phases succeed one another. The first covers the algorithmic type of algebra dealing with numerical values or a non-symbolic model, e.g. Greek geometrical algebra. The second is home to algebras employing words or abbreviations for the unknown but not therefore being symbolic in character, such as Diophantan and early Abbacus algebra. The third and last phase, that of (truly) symbolic algebra, viz. allowing for manipulations on the symbolic level of symbols only, starts around 1560.

Ad Meskens (Antwerp) in his contribution explains that Diophantos had complete mastery over the methods of solution for solving linear equations, for indeterminate quadratic equations and for systems of equations of first and second degree, while for higher degree equations he only sometimes had a solution method. When in 1971 an Arabic version of Diophantos was found, it came as a shock that in this book some previously unknown material was found. In it, Diophantos apparently used the methods for solving higher-degree problems described in other books to their limits. The structure of a Diophantine problem follows a general rule set out by Proklos. Problems are put a general way, using indeterminate numbers. But solutions are provided for specific numbers given at the outset. We can see a partial analogy with geometrical construction: applied to a specific figure, though posed generally. Even if the specific example allows it, Diophantos never gives general methods. During the elaboration of the example he sometimes adds a restriction. In some other cases, where there is a need for restrictions, he does not impose them. It is unclear whether this is due to ignorance about the need of such a restriction or to the impossibility to correctly formulate it.

According to *Jens Høyrup* (Roskilde), Italian fourteenth- and fifteenth-century abbacus algebra presents us with a number of deviations from what we would consider normal mathematical practice and proper mathematical behaviour: the invention of completely false algebraic rules for the solution of cubic and quartic equations, and of rules that pretend to be generally valid but in fact only hold in very special cases; and (in modern terms) an attempt to expand the multiplicative semi-group of non-negative algebraic powers into a complete group by identifying roots with negative powers. In both false-rule cases, the authors of the fallacies must have known they were cheating. Certain abbacus writers seem to have discovered, however, that something was wrong, and devised alternative approaches to the cubics and quartics; they also developed safeguards against the misconceived extension. In his paper, Høyrup analyses both phenomena, and correlates them with the general practice and norm system of abbacus mathematics as this can be extracted from the more elementary level of the abbacus treatises.

Matthew Parker's (London) contribution considers Cantor's extension of the concept of number to the transfinite, and the resolution this supplies for what has been called "Galileo's Paradox", namely that the square numbers seem to be at once fewer than and equal to the positive integers. Galileo's Paradox is held to have been resolved by the articulation of numerosity into distinct concepts, including those of proper inclusion, *Anzahl*, and power. Power has become the basis of an elegant and useful theory and has proven especially useful in addressing the motivations common to Galileo, Bolzano and Cantor, namely, to grasp the relations between numerosity and geometric magnitude, to defend the analysis of the continuum into points, and to explain physical phenomena. As Parker explains, it is in virtue of its success in serving such motivations that Cantor's theory of transfinite numbers constitutes a solution to some of the deeper philosophical problems posed by Galileo's Paradox. But there are alternatives. For example, *Anzahl* too can be considered as a notion of numerosity. In order to analyze this matter, Parker proposes a Method of Conceptual Articulation.

During the first part of the nineteenth century, the mathematical disciplines of analysis and algebra developed tremendously, exploring new techniques and questions to arrive at far-reaching and sometimes surprising new results. According to *Henrik Kragh Sørensen* (Aarhus), a central part of this development involved changes in the role and use of representations. Mathematicians often work with representations in order to access and manipulate mathematical objects such as functions. These uses can satisfy a variety of demands. For instance, representations of implicitly defined functions as infinite series can add to the familiarity of these new functions by anchoring them within existing ways of accessing and manipulating functions. In different contexts, the question of whether a given function can or cannot be represented in a specific form opens the door for new results such as impossibility proofs. In his paper, Kragh Sørensen analyses such multiple roles of representations of functions from a Wittgenstein-inspired perspective as "aspects" of functions. By comparing important results from algebra (algebraic unsolvability of the general quintic equation) and analysis (representations of elliptic functions) in the context of Abel's mathematics, representations are highlighted both as means and ends in themselves.

Point of departure of the next article, by *Jeremy J. Gray* (Milton Keynes), is the observation that, on the one hand, historians and philosophers of mathematics share an interest in the nature of mathematics (what it is, what features affect its growth, how it informs other disciplines), but that, on the other hand, much of the work done in history and philosophy of mathematics shows that the two groups largely work in isolation. A reconsideration of the history of mathematical analysis in the nineteenth century, according to Gray, suggests that history and philosophy of mathematics can be done together to the advantage of both, and also how legitimately different enquiries need not drive them apart.

The last few decades have witnessed a broadening of the philosophy of mathematics, beyond narrowly foundational and metaphysical issues, and towards the inclusion of more general questions concerning methodology and practice. Part of this broadening, although a part that remains relatively close to foundational and metaphysical issues, is the turn towards a "new epistemology" for mathematics, including the study of topics such as the role of visualization in mathematics, the use of computers in proving mathematical theorems, and the notion of explanation as applied to mathematics. *Erich H. Reck*'s (Riverside, CA) paper is a contribution to such a new epistemology. More particularly, it is an attempt to bring into sharper focus, and to argue for the relevance of, two related themes: "structural reasoning" and "mathematical understanding". As the notion of understanding is vague and slippery in general, as well as very loaded in philosophical discussions of the sciences, the label is handled with care. Similarly, while talking about "structural" reasoning in mathematics may be suggestive, that term too requires further elaboration. Reck's clarifications and elaborations are tied to a specific historical figure and period, Richard Dedekind, and his contributions to algebraic number theory in the nineteenth century, which proves to be all but an incidental choice.

Eduard Glas (Delft) undertakes a comparison between mathematician Felix Klein and philosopher Imre Lakatos. Klein, Glas argues, is perhaps the most outstanding example of an eminently fruitful mathematician opposing the one-sided obsession of most mathematicians of his generation with purity and rigor, an obsession through which the discipline increasingly tended to fall apart into disparate, self-contained specialties. In contrast to the adepts of rigor and purity within the leading schools, who eschewed reliance on intuitive or quasi-empirical insights, Klein's methodology was based on the use of geometric and even physical models and thought experiments, a methodology which certainly qualifies as 'quasi-empirical'. Klein's successes depended in large measure on his exceptional versatility in the mental visualisation even of the most abstract mathematical objects and relations. Throughout his career, Klein kept insisting that intuition, especially spatial intuition, is indispensable in all mathematical endeavours, which also makes for their rootedness in concrete experience. According to Glas, Klein was as much a maverick in the eyes of 'pure' mathematicians as Imre Lakatos would become in the eyes of mainstream philosophers of mathematics. Like Lakatos, Klein insisted that progress in mathematics relies on methods that are very much akin to those of natural science, especially as concerns the use of models and (thought) experiments. He in fact practised a model-based, quasi-empirical method of investigation that indeed tallies nicely with Lakatos' quasi-empiricist methodology.

The idea that formal geometry derives from intuitive notions of space has appeared in many guises, most notably in Kant's argument from geometry. Kant claimed that a priori knowledge of spatial relationships both allows and constrains formal geometry: it serves as the actual source of our cognition of the principles of geometry and as a basis for its further cultural development. The development of non-Euclidean geometries, however, undermined the idea that there is some privileged relationship between our spatial intuitions and mathematical theory. The aim of the paper by *Helen De Cruz* (Leuven) is to look at this longstanding philosophical issue through the lens of cognitive science. Drawing on recent evidence from cognitive ethology, developmental psychology, neuroscience and anthropology, she argues for an enhanced, more informed version of the argument from geometry: humans share with other species evolved, innate intuitions of space which serve as a vital precondition for geometry as a formal science.

In line with the general spirit of the underlying conference, *Ronny Desmet* (Brussels) observes that it is part of the growing awareness that historical, social and psychophysical processes precede the cut and dried results of mathematics, even those which have been presented as the obvious starting points of all pure mathematics. And as with all fashionable currents, he continues, the shift from foundations to practices in the philosophy of mathematics has its heroes. Indeed, Lakatos and Wittgenstein immediately come to mind in this respect. If an author were to say that, given their mutual influence, Wittgenstein's view on mathematics can be identified with Russell's, dissent would follow. But if he were to say that, given their intense collaboration Whitehead's view on mathematics can be identified with Russell's, this claim would normally pass without much protest. Desmet, however, could not disagree more. In his paper, he argues that Whitehead, like Wittgenstein, should be differentiated from Russell, and given his own niche in philosophy of mathematics, and that, furthermore, Whitehead's writings, which were based on his own mathematical experience, offer a perspective on mathematical practices equalling, or even surpassing, that provided by Wittgenstein.

In discussions of mathematical practice, *Dirk Schlimm* (Montréal) points out, the role axiomatics has often been confined to providing the starting points for formal proofs, with little or no effect on the discovery or creation of new mathematics. Nevertheless, it is undeniable that axiomatic systems have played an essential role in a number of mathematical innovations. Moreover, it was not only through the investigation and modification of given systems of axioms that new mathematical notions were introduced, but also by using axiomatic characterizations to express analogies and to discover new ones. In his contribution, which closes this volume, Schlimm however draws our attention to a different use of axiomatics in mathematical practice, namely that of being a vehicle for bridging theories belonging to previously unrelated areas. How axioms have been instrumental in linking mathematical theories is illustrated by the investigations of Boole, Stone, and Tarski, all of which revolve around the notion of Boolean algebra.

As already mentioned, a companion volume of (more philosophically oriented) PMP2007 proceedings papers is to be published by College Publications, London. At the outset of the present one, allow me to express my gratitude to an array of people. For naturally, this proceedings volume did not come about because of the efforts of the editor alone. To begin with, I am extremely grateful for all institutional and personal contributions to what in my eyes was a very successful PMP2007 conference. This includes the generous sponsors of the event: Research Foundation — Flanders, Brussels Capital-Region, National Centre for Research in Logic — Belgium, as well as its (co-)organizers Belgian Society for Logic and Philosophy of Science, Wissenschaftliches Netzwerk PhiMSAMP, and Centre for Logic and Philosophy of Science at Free University of Brussels. Further, I was very fortunate to leave local organization largely in the safe hands of my precious colleagues Patrick Allo, Ronny Desmet, and Karen François. And as there is simply no event whatsoever without an interested and interesting audience, let me hereby also thank all participants to PMP2007 (including the authors of this book). I sincerely hope that this conference series, started in 2002 with the initial PMP, may live on, possibly at other locations. On a more personal level, I also feel indebted to the bodies that have funded my academic work during the past years: Research Foundation — Flanders, Free University of Brussels (through BAP and GOA49), and Alexander von Humboldt-Foundation — Germany (with special thanks to my dear colleague and friend Thomas Müller). Finally, I would love to dedicate this volume to Jean Paul Van Bendegem, who has been my faithful and trusting mentor for a decade now, and to whom, philosophically but also beyond, I owe so much.

Bart Van Kerkhove
(editor)

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ON THE NATURE AND ORIGIN OF ALGEBRAIC SYMBOLISM

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1. The myth of syncopated algebra

Ever since Nesselmann's study on "Greek algebra" (1842),[1] historical accounts on algebra draw a distinction between rhetorical, syncopated and symbolic algebra. This tripartite distinction has become such a commonplace depiction of the history of algebraic symbolism that modern-day authors even fail to mention their source (e.g., Boyer,[2] p. 201; Flegg and Hay;[3] Struik[4]). The repeated use of Nesselmann's distinction in three *Entwickelungstufen* on the stairs to perfection is odd because it should be considered a highly normative view which cannot be sustained within our current assessment of the history of algebra. Its use in present-day text books can only be explained by an embarrassing absence of any alternative models. There are several problems with Nesselmann's approach.

1.1. *A problem of chronology*

Firstly, if seen as steps within a historical development, as is most certainly the view shared by many who have used the distinction, it suffers from some serious chronological problems. Nesselmann[1] (p. 302) places Iamblichus, Arabic algebra, Italian abbacus algebra and Regiomontanus under rhetorical algebra ("Die erste und niedrigste Stufe") and thus covers the period from 250 to 1470. A solution to the quadratic problem of al-Kwārizmī is provided as an illustration. The second phase, called syncopated algebra, spans from Diophantus's *Arithmetica* to European algebra until the middle of the seventeenth century, and as such includes Viète, Descartes and van Schooten. Nesselmann discusses problem III.7 of the *Arithmetica* as an example of syncopated algebra. The third phase is purely symbolic and constitutes modern algebra with the symbolism we still use today. Nesselmann repeats the example of al-Kwārizmī in modern symbolic notation to illustrate the third phase, thereby making the point that it is not the procedure or contextual elements but the use of symbols that distinguishes the three phases.

Though little is known for certain about Diophantus, most scholars situate the *Arithmetica* in the third century which is about the same period as Iamblichus (c. 245-325). So, syncopated algebra overlaps with rhetorical algebra for most of its history. This raises serious objections and questions such as "Did these two systems influence each other?" Obviously, historians as Tropfke[5] (II, p. 14) and Gandz[6] (p. 271) were struck by this chronological anomaly and formulated an explanation. They claim that Arabic algebra does not rely on Diophantus' syncopated algebra but descends instead from Egyptian and Babylonian problem-solving methods which were purely rhetorical. However, these arguments are now superseded by the discovery of the Arabic translations of the *Arithmetica* (Sesiano[7] and Rashed[8]). Diophantus was known and discussed in the Arab world ever since Qustā ibn Lūqā (c. 860). So if the syncopated algebra of Diophantus was known by the Arabs, why did it not affect their rhetorical algebra? If the Greek manuscripts, used for the Arab translation of the *Arithmetica* contained symbols, we would expect to find some traces of it in the Arab version.

1.2. *The role of scribes*

The earliest extant Greek manuscript, once in the hands of Planudes and used by Tannery,[9] is Codex Matritensis 4678 (ff. 58-135) of the thirteenth century. The extant Arabic translation published independently by Sesiano[7] and Rashed[8] was completed in 1198. So no copies of the *Arithmetica* before the twelfth century are extant. Ten centuries separating the original text from the earliest extant Greek copy is a huge distance. Two important revolutionary changes took place around the ninth century: the transition of papyrus to paper and the replacement of the Greek uncial or majuscule script by a new minuscule one. Especially the transition to the new script was a drastic one. From about 850 every scribe copying a manuscript would almost certainly adopt the minuscule script (Reynolds and Wilson,[10] pp. 66–7). Transcribing an old text into the new text was a laborious and difficult task, certainly not an undertaking to be repeated when a copy in the new script was already somewhere available. It is therefore very likely that all extant manuscript copies are derived from one Byzantine archetype copy in Greek minuscule. Although contractions were also used in uncial texts, the new minuscule much facilitated the use of ligatures. This practice of combining letters, when performed with some consequence, saved considerable time and therefore money. Imagine the time savings by consistently replacing *ἀριθμος*, which appears many times for every problem, with ς in the whole of the *Arithmetica*. The role of professional scribes should therefore not be underestimated. Although we find some occurrences of shorthand notations in papyri, the paleographic evidence we now have on a consistent use of ligatures and abbreviations for mathematical words points to a process initiated by mediaeval scribes much more than to an invention by classic Greek authors. Whatever syncopated nature we can attribute to the *Arithmetica*, it is mostly an unintended achievement of the scribes.[a] The complete lack of any syncopation in the Arabic translation further supports this thesis. The name for the unknown and the powers of the unknown and even numbers are written by words in Arabic translation. The lack of, at that time, well-established Hindu-Arabic numerals seems to indicate that the Arabic translation was faithful to a Greek majuscule archetype. Sesiano[7] (p. 75) argues that the Arabic version relies on the commentary by Hypatia while the Greek versions relate to the original text with some early additions and interpolations. Although the thesis of the reliance on Hypatia's commentaries is strongly opposed by Rashed[8] (III, p. LXII), and while they disagree on many others issues, both interpretations and translations of the Arabic text concur on the lack of symbolism or syncopation. The *αλογος αριθμος*, or 'untold number' of the Greek text, is translated as *śay'* in Arab, and is thus very similar to the *cosa* of abbaco texts or the *coss* of German cossic texts.

In so far the *Arithmetica* deserves the special status of syncopated algebra, it is very likely that the use of ligatures in Greek texts is a practice that developed since the ninth century and not one by Diophantus during the third century. This overthrows much of the chronology as proposed by Nesselmann.

1.3. *Symbols or ligatures?*

A third problem concerns the interpretation of the qualifications 'rhetorical' and 'syncopated'. Many authors of the twentieth century attribute a

[a] This view also has recently been put forward in relation to Archimedes' works (Netz and Noel[11]).

highly symbolic nature to the *Arithmetica* (e.g. Kline,[12] I, pp. 139–40). Let us take Cajori[13] (I, pp. 71–4) as the most quoted reference on the history of mathematical notations. Typical for Cajori's approach is the methodological mistake of starting from modern mathematical concepts and operations and looking for corresponding historical ones. He finds in Diophantus no symbol for multiplication, and addition is expressed by juxtaposition. For subtraction the symbol is an inverted ψ. As an example he writes the polynomial

$$x^3+13x^2+5x+2 \quad \text{as} \quad \mathrm{K}^Y\bar{\alpha}\Delta^Y\bar{\iota\gamma}\varsigma\bar{\epsilon}\mathring{M}\bar{\beta}$$

where K^Y, Δ^Y, ς are the third, second and first power of the unknown and $\mathring{M}$ represents the units. Higher order powers of the unknown are used by Diophantus as additive combination of the first to third powers.

Cajori makes no distinction between symbols, notations or abbreviations. In fact, his contribution to the history of mathematics is titled *A History of Mathematical Notations.* In order to investigate the specific nature of mathematical symbolism one has to make the distinction somewhere between symbolic and non-symbolic mathematics. This was, after all, the purpose of Nesselmann's distinction. We take the position together with Heath,[14] Ver Eecke[15] and Jacob Klein, that the letter abbreviations in the *Arithmetica* should be understood purely as ligatures (Klein,[16] p. 146):

> We must not forget that *all* the signs which Diophantus uses are merely word abbreviations. This is true, in particular for the sign of "lacking", $\pitchfork$, and for the sign of the unknown number, ς, which (as Heath has convincingly shown) represents nothing but a ligature for $\alpha\varrho$ ($\acute{\alpha}\varrho\iota\theta\mu o\varsigma$).

Even Nesselmann[1] acknowledges that the 'symbols' in the *Arithmetica* are just word abbreviations ("sie bedient sich für gewisse oft wiederkehrende Begriffe und Operationen constanter Abbreviaturen statt der vollen Worte"). In his excellent French literal translation of Diophantus, Ver Eecke[15] consequently omits all abbreviations and provides a fully rhetorical rendering of the text as in "Partager un carré proposé en deux carrés" (II.8), which makes it probably the most faithful interpretation of the original text.[b]

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[b] This problem led Fermat to add the marginal note in his copy of Bachet's translation "Cubum autem in duos cubos, aut quadratoquadratum in duos quadratoquadratos, et

This objection marks our most important critique on the threefold distinction: symbols are not just abbreviations or practical short-hand notations. Algebraic symbolism is a sort of representation which allows abstractions and new kinds of operations. This symbolic way of thinking can use words, ligatures or symbols, as we will argue further. The distinction between words, word abbreviations and symbols is in some way irrelevant with regards to the symbolic nature of algebra.

1.4. *Counter-examples*

A final problem for Nesselmann's tripartite distinction is that now, almost two centuries later, we have a much better understanding of the history of symbolic algebra. Nesselmann relied mostly on the Jesuit historian Cossali[17] for a historical account of Italian algebra before the sixteenth century. Except for a text by Rafello Canacci, Cossali does not discuss much the algebra as it was practiced within the abbacus tradition of the fourteenth and fifteenth century. Guillaume Libri, who had collected many manuscripts from this tradition, describes and published several transcriptions in his *Histoire des sciences mathématiques en Italie* published in 1838.[18] Oddly, the well-informed Nesselmann does not seem to know Libri's work and thus remains ignorant of the continuous practice of algebra in Italy since Fibonacci and the first Latin translations of al-Kwārizmī. It is only since the last few decades that we have a more complete picture of abbacus algebra, thanks to the work of Gino Arrighi, Warren van Egmond, the *Centro studi della matematica medioevale* of Sienna and Høyrup's recent book (Høyrup[19]). In our understanding, symbolic algebra is an invention of the sixteenth century which was prepared by the algebraic practice of the abbacus tradition. At least abbacus algebra has to be called syncopated in the interpretation of Nesselmann. A lot of abbacus manuscripts use abbreviations and ligatures for *cosa*, the unknown (as c, co. or ϱ), censo or cienso, the second power of the unknown (ce. or ç), cubo, the third power (cu.) and beyond. Also plus, minus and the square root are often abbreviated, such as in p, m and R (with an upper or lower dash). From the fifteenth century we also find manuscripts that explicitly refer to a method of solving problems that is different from the regular rhetorical method. In an anonymous manuscript of c. 1437,[c] the author solves several standard problems in two ways. One

generaliter nullam in infinitum ultra quadratum potestatem in duos eiusdem nominis fas est dividere". If Fermat had used the 'syncopated' algebra of Diophantus he might have had some marginal space left to add his "marvelous proof" for this theorem.

[c] Florence, Biblioteca Nazionale, Magl. Cl. XI. 119. See Heeffer[20] for a critical edition.

he calls symbolical (*figuratamente*) and the other rhetorical (*per scrittura*). Possibly the practice of solving a problem *figuratamente* existed before, but in any case it was not found suitable to be written down in a treatise. Here however, the anonymous author believes a symbolic notation contributes to a better understanding of the solution as he writes:

> I showed this symbolically as you can understand from the above, not to make things harder but rather for you to understand it better. I intend to give it to you by means of writing as you will see soon.[d]

He then repeats the solution in a rhetorical form as we know it from other abbacus texts. This is the first occasion in the history of algebra where an author makes an explicit reference to two different kinds of problem solving, which we would now call symbolical and non-symbolical.

This manuscript or related copies may have influenced the German cossists. Regiomontanus, who maintained close contacts with practitioners of algebra in Italy, adopts the same symbolic way of solving problems. In his correspondence with Johannes Bianchini of 1463 we find problems very similar to the abbacus text: divide 10 into two parts so that one divided by the other together with the other divided by the first equals 25.[e] In modern symbolic notation the problem can be formulated as in Fig. 1:

Regiomontanus solves the problem in the same manner of abbacus algebra but adopts only the symbolical version. He uses symbols for *cosa* and *censo* which we typically find in German cossist algebra from 1460 for a period of about 160 years.

While we see in later abbacus algebra and Regiomontanus the roots of symbolic algebra, Nesselmann places both within the stage of rhetorical algebra. According to Nesselmann's own definition these two instances of algebraic practice should at least be called syncopated.

[d] f. 59r: "Ora io telo mostrata figuratuiamente come puoi comprendere di sopra bene che e lla ti sia malagievole ma per che tulla intenda meglio. Io intende di dartela a intendere per scrittura come apresso vedrai".

[e] The correspondence is kept in Nürnberg, City Library, Cent. V, 56c, ff. 11r-83v, The transcription is by Curtze[21] (pp. 232–4): "Divisi 10 in duos, quorum maiorem per minorem divis, item minorem per maiorem. Numeros quotiens coniunxi, et fuit summa 25 : quero, que sint partes". The corresponding problem in Magl. Cl. XI. 119 is on f. 61v but uses a sum of 50 instead of 25.