Topological Measures and Weighted Radon Measures

D.P.L. Castrigiano
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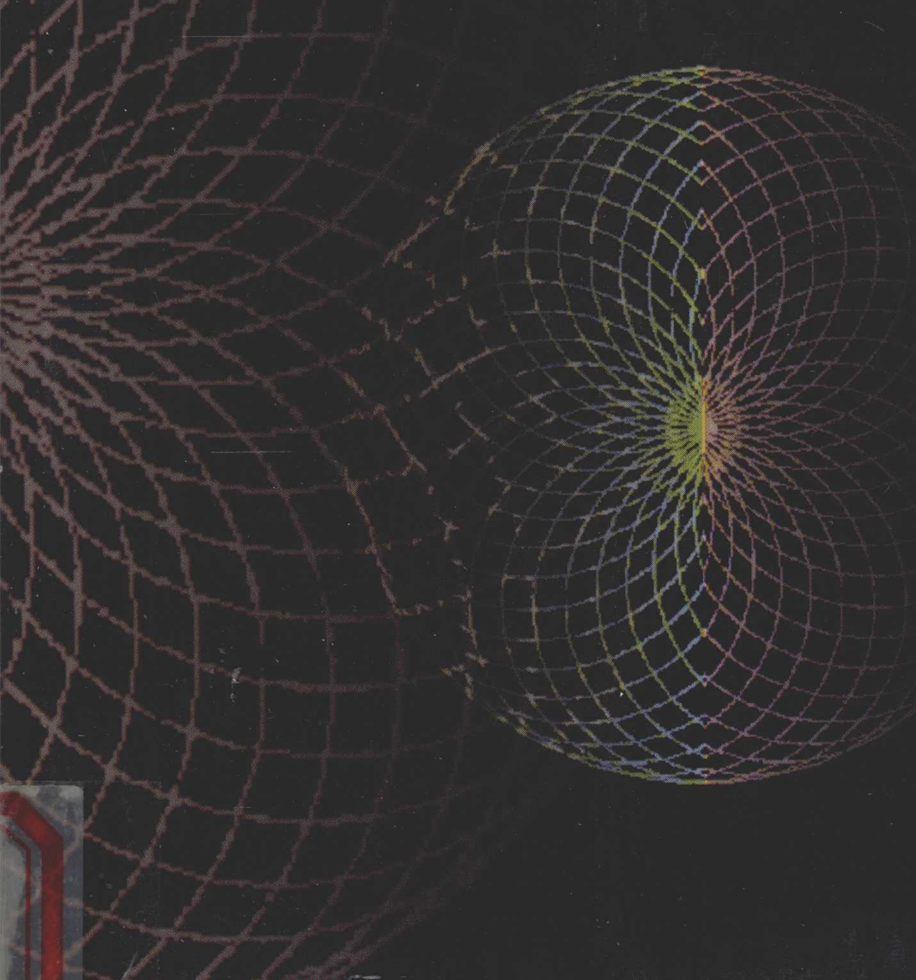

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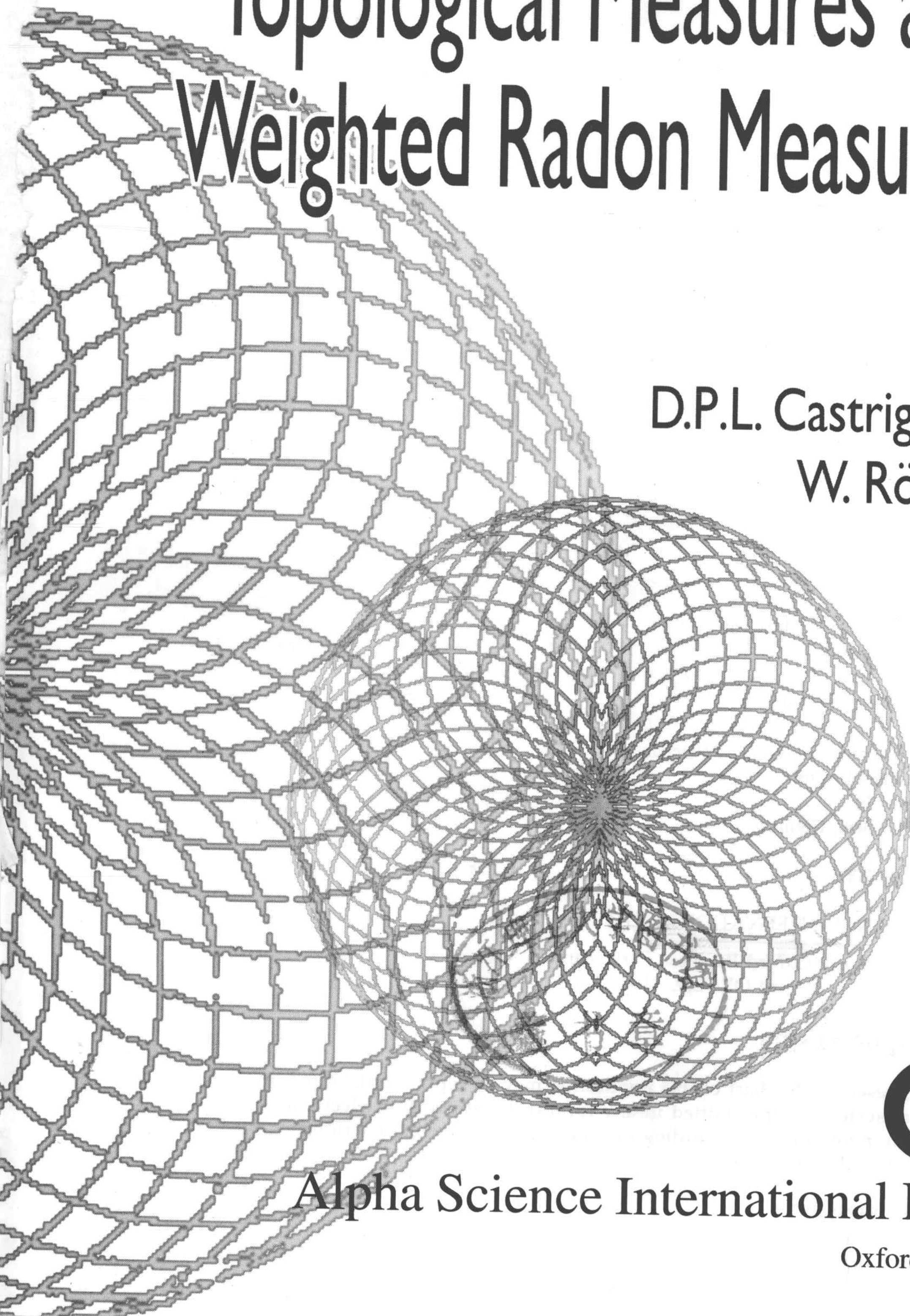

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Preface

Topological measure means a measure on a σ-algebra of subsets of a topological Hausdorff space which contains the open sets. Hence a topological measure is defined at every Borel set of the underlying Hausdorff space. This book provides a detailed exposition of the theory of topological measures. The measure theoretical prerequisites are furnished in the first chapter.

Topological measure theory is a particularly intriguing part of general measure theory. Its special attraction is due to the close interplay of measure and topology. The connection between these two disciplines is provided by the regularity properties which the topological measures assume with respect to the topology of the underlying Hausdorff space. Recall as classic examples of such measures the Lebesgue measure on the Euclidean spaces $\mathbb{R}^n$ and more generally the Haar measures on locally compact topological groups. Many of the usual regularity properties are treated throughout the book. We add two new regularity properties. The notion of local tightness is relevant to the study of pre-Radon spaces leading in the end to a certain class of Radon spaces. The mc property proves very useful for the study of weighted Radon measures.

Radon measure denotes a topological measure which is locally finite and inner and outer regular. Radon measures correspond one-to-one to locally finite tight measures which we treat as well throughout the book.

Roughly speaking Chapter 2 is concerned with the interdependencies among the various regularity properties as well as the implications of these properties on other properties of topological measures. Particularly interesting are the implications of topological properties of the underlying space on the regularity of the topological measures. A main object of topological measure theory, with which we are concerned, is to determine those topological spaces, called Radon spaces, on which every finite Borel measure is a Radon measure. In this context we introduce locally tight spaces. These are pre-Radon spaces, in which the locally K-analytic spaces are prominent. Another topic which we study under various premises is the Borel regularity of the essential outer measure of a topological measure. The usual case is that the essential outer measure of a Radon measure is Borel regular. We have some positive results, e.g. when the underlying space is weakly Θ-refinable. However, in general, a non-tight Radon measure does not need to share this property, as an example on a σcc space and on a locally compact space shows. Assuming CH there is also a negative example for a tight Radon measure on a locally compact space. It turns out that within ZFC it is not possible to decide whether there exists a tight Radon measure on a regular space with non-Borel regular essential outer measure.

The considerations on topological measures in Chapter 2, which contain also several results on integration theory, are preceded by a thorough discussion in Chapter 1 of the outer measure and the essential outer measure and their respective integrals of an abstract measure. This exposition includes also measures multiplied by a density function and is interesting by itself. The main purpose we pursue by this introductory chapter is to determine clearly those parts of the results on topological measures which are not of purely measure theoretical origin.

Weighted Radon measures are obtained by multiplying a Radon measure by a locally integrable non-negative extended real-valued function. This is an as elementary as useful construction of measures, which arises in applications as well as in theory, but which does not preserve regularity properties. In general, neither inner nor outer regularity of a Radon measure is maintained. Regularity of a weighted Radon measure obviously depends on the weight function, the Radon measure itself, and the topology of the underlying Hausdorff space. It seems that literature does not deal much with this attractive topic. We study the involved dependences to some extent in Chapter 3. Things become remarkably clear if both measures, i.e., the Radon measure and the corresponding weighted measure, have the mc property. In this case the regularity of the weighted Radon measure essentially depends only on the zero set of the weight function in relation to the Radon measure. This favorable situation occurs automatically if the underlying Hausdorff space is Lc, which means that the closure of every σ-compact subset is a Lindelöf set. If the closure is even σ-compact, then the space is called σcc.

The content of Chapter 4 is purely topological. It is devoted to the study of σcc and Lc spaces. The question is treated how the σcc and Lc properties are related to other topological properties. For instance, it follows that paracompact spaces or generalized ordered spaces are Lc, and P-spaces are σcc. On the other hand several types of locally compact spaces are constructed with many additional topological properties, which are not σcc. Also some general results on mappings, coverings, and products are obtained concerning the σcc and Lc properties.

We found it particularly important to provide examples and counterexamples commenting on the results. The last chapter is a collection of (counter)examples which arises from the discussion of the results on the regularity of topological measures, in particular of weighted Radon measures, obtained in previous chapters. Typically the sufficiency and necessity of the premises as well as the possibility to strengthen the results are examined. Several interesting examples are based on the infinite topological product of probability measures on compact spaces, which is treated in the last section of the second chapter. The prior section of that chapter is concerned with two special types of topological measure spaces from which many other examples are taken. Similarly, the last section of Chapter 4 is devoted to a number of (counter)examples illustrating and completing the previous results on the σcc and Lc properties. But also throughout the remaining parts of the book we included many elucidating examples.

The level of the presentation of the material is advanced undergraduate. It presupposes some familiarity with elementary measure and integration theory and assumes little beyond the basic definitions and results from set theoretic topology. Only at some points in the chapters 2, 4, and 5 more set-theoretic skill is required. As to the references, we cite the literature which was near at hand and made no effort to trace the results to the origins.

We are indebted to W. Adamski, W.W. Comfort, D.H. Fremlin, P. Nyikos, V. Uspenskij, and J.V. Yascenko for providing useful information.

Contents

Chapter 1

Abstract measures and densities

We collect and develop some facts from the theory of abstract measures and integration, much of which will be used later. Tacitly we refer to elementary measure and integration theory, see e.g. [Bau ; Kap. I, II], [Els ; Kap. I-IV], or [HewStr ; §10-12].

Throughout this chapter, let $(X, \mathcal{B}, \mu)$ be a **measure space**, i.e., X a set, $\mathcal{B}$ a σ–algebra on X, and μ a measure on $\mathcal{B}$. We recall that $\mathcal{B}$ is a non-void set of subsets of X being closed under complementation and countable unions, and μ is a non-negative extended real-valued countably additive (i.e. σ-additive) function on $\mathcal{B}$ being zero at the empty set $\emptyset$. The sets $A \in \mathcal{B}$ with $\mu(A) = 0$ are called **μ–null sets**. A function f on X is said to be **μ–integrable** if it is $\mathcal{B}$–measurable with $\int |f| \, d\mu < \infty$. Recall also that an **outer measure** ω on a set X is a non-negative extended real-valued function on the **power set** $\mathcal{P}(X)$ of X which is monotone countably subadditive (i.e. σ-subadditive) and vanishes at the empty set. The sets $A \subset X$ with $\omega(A) = 0$ are called **ω–null sets**, and a set $A \subset X$ is called **ω–σ–finite** if $A \subset \bigcup_{n=1}^{\infty} A_n$ with $\omega(A_n) < \infty$.

Generally, we use standard notation. $\mathbb{N}$ denotes the set of positive integers. For extended real-valued functions f and f_n, $n \in \mathbb{N}$, the notation $f_n \uparrow f$ means $f_1 \le f_2 \le \ldots$ and $f_n \to f$ pointwisely. Also $f > 0$ is understood pointwisely.

1.1 Outer measure and upper integral

The **outer measure μ^* on X associated with μ** is given by

(1) $$\mu^*(A) := \inf\{\mu(B) : B \supset A,\ B \in \mathcal{B}\}, \quad A \subset X.$$

As suggested the set function μ^* is an outer measure, see (10). Clearly, μ^* extends μ and the infimum in (1) is attained. — One should note that, in general, $\mu^*(B \setminus A) > 0$ holds for every $B \in \mathcal{B}$ with $B \supset A$, even if μ is finite. Actually, $\mu^*(B \setminus A) = 0$ for some such B implies $A \in \hat{\mathcal{B}}$, cf. (25). — If some B in (1) is σ-finite, then there is a $C \supset A$, $C \in \mathcal{B}$ such that $\mu(C \setminus B) = 0$ for all B in (1). Indeed, let A be covered by countably many $E_n \in \mathcal{B}$ of finite measure. Choose $C_n \in \mathcal{B}$ with $C_n \supset A \cap E_n$ and $\mu(C_n) = \mu^*(A \cap E_n)$. This obviously implies $\mu(C_n \setminus B) = 0$. Set $C := \bigcup_n C_n$.

More generally one defines the **upper integral** $\int^* f\,d\mu$ for any function $f : X \to [0,\infty]$, shortly $f \geq 0$, by

(2) $$\textstyle\int^* f\,d\mu := \inf\{\int s\,d\mu : s \geq f,\ s \text{ step function}\},$$

where a **step function** s on X takes only countably many values $\alpha_n \in [0,\infty]$ such that $s^{-1}(\{\alpha_n\}) \in \mathcal{B}$, and where its integral (in the ordinary sense) is $\int s\,d\mu = \sum_n \alpha_n \mu(s^{-1}(\{\alpha_n\}))$. Here, as throughout measure theory, $0 \cdot \infty = \infty \cdot 0 = 0$ holds. For $A \subset X$ let 1_A denote the **indicator function** of A. Then, obviously,

(3) $$\textstyle\mu^*(A) = \int^* 1_A\,d\mu.$$

(4) **Proposition.** *The upper integral extends the integral, i.e.,*

$$\textstyle\int^* f\,d\mu = \int f\,d\mu \quad \textit{for all } \mathcal{B}\textit{-measurable } f \geq 0.$$

Proof. Note first that $\int f\,d\mu \leq \int^* f\,d\mu$ holds by the definition of $\int^*$. Therefore one may assume $\int f\,d\mu < \infty$. Then $\{f > 0\}$ is σ–finite. Hence $\{f > 0\}$ is the union of countably many, mutually disjoint $A_n \in \mathcal{B}$, each of finite measure. It suffices to show (4) for finite measure, since then, if $\varepsilon > 0$, there are step functions $s_n \geq 1_{A_n} f$ satisfying $\int 1_{A_n} s_n\,d\mu \leq \int 1_{A_n} f\,d\mu + 2^{-n}\varepsilon$, whence $\int s\,d\mu \leq \int f\,d\mu + \varepsilon$ for the step function $s := \sum_n 1_{A_n} s_n \geq f$.

In addition, an analogous argument reduces (4) to the case that f is bounded, since $f = \sum_n 1_{\{n-1<f\leq n\}} f + 1_{\{f=\infty\}} f$.

Now let μ be finite and let $f \leq \alpha 1_X$ for some $\alpha \in [0,\infty[$. For $\varepsilon > 0$ there is an elementary function $u \leq \alpha 1_X - f$ with $\int(\alpha 1_X - f)\,d\mu \leq \int u\,d\mu + \varepsilon$. Then $s := \alpha 1_X - u$, which of course is a step function, satisfies $f \leq s$ and $\int s\,d\mu \leq \int f\,d\mu + \varepsilon$. This implies the assertion. □

If $f \geq 0$ is $\mathcal{B}$-measureable, then the step functions

(5) $$s_k := 2^{-k} \sum_{n=1}^{\infty} 1_{\{f>n2^{-k}\}} \quad \textit{for} \;\; k \in \mathbb{N}$$

satisfy $f \leq s_k + 2^{-k}$ and $s_k \uparrow f$, and the monotone convergence theorem yields $\int f\,d\mu = \lim_k \int s_k\,d\mu$. Hence, by (4), f is sandwiched between upper and lower step functions whose integrals tend to $\int f\,d\mu$.

(6) **Proposition.** *The upper integral is monotone, countably subadditive, and satisfies the Fatou property , i.e.,*

(7) $$0 \leq f \leq g \quad \Rightarrow \quad 0 \leq \int^* f\,d\mu \leq \int^* g\,d\mu,$$

(8) $$f_n \geq 0,\, n \in \mathbb{N} \quad \Rightarrow \quad \int^* \left(\textstyle\sum_n f_n\right) d\mu \leq \textstyle\sum_n \int^* f_n\,d\mu,$$

(9) $$f_n \geq 0, f \geq 0,\, f_n \uparrow f \quad \Rightarrow \quad \int^* f_n\,d\mu \uparrow \int^* f\,d\mu.$$

Proof. Monotonicity is obvious. — To show (8), let $s_n \geq f_n$ be a step function. Then $s := \sum_n s_n$ is a step function satisfying $s \geq f := \sum_n f_n$. Because of the countable additivity of the integral, $\sum_n \int s_n\,d\mu = \int s\,d\mu \geq \int^* f\,d\mu$, and therefore, $\int^* f\,d\mu \leq \inf\{\sum_n \int s_n\,d\mu \,:\, s_n \geq f_n, n \in \mathbb{N}\} = \sum_n \inf\{\int s_n\,d\mu \,:\, s_n \geq f_n\} = \sum_n \int^* f_n\,d\mu$. — To show the Fatou property (9), let $s_n \geq f_n$ be step functions such that $\int s_n\,d\mu \leq \int^* f_n\,d\mu + \frac{1}{n}$. Then $g_n := \inf\{s_m \,:\, m \geq n\}$ is $\mathcal{B}$-measurable and satisfies $s_n \geq g_n \geq f_n$, since $s_m \geq f_m \geq f_n$ for all $m \geq n$, and therefore $\int^* f_n\,d\mu \leq \int g_n\,d\mu \leq \int s_n\,d\mu \leq \int^* f_n\,d\mu + \frac{1}{n}$ by (4). The first and the last term tend to $a := \sup_n \int^* f_n\,d\mu$, and $\int g_n\,d\mu$ tends to $\int g\,d\mu$ where $g := \sup_n g_n = \liminf_n s_n$ because of the Fatou property of integrals, i.e. B. Levi's theorem [HewStr ;(12.22)]. Therefore $a = \int g\,d\mu \geq \int^* f\,d\mu \geq a$ by (4) since $g \geq f \geq f_n$. □

(10) **Corollary.** *The set function μ^* is an outer measure having the Fatou property, i.e.,*

$$A_n,\, A \subset X,\, A_n \uparrow A \quad \Rightarrow \quad \mu^*(A_n) \uparrow \mu^*(A).$$

Proof. This follows from (3), (6), and since $\mu^*(\emptyset) = 0$. □

Note that not all outer measures have the Fatou property. As a simple **example** consider the outer measure ω on $X = \mathbb{N}$ given by $\omega(\emptyset) := 0$, $\omega(A) := 1$ if A is not empty and finite, and $\omega(A) := \infty$ if A is infinite.
From (2), and using (9) if $\alpha = \infty$, it follows easily

(11) $\int^* \alpha f \, d\mu = \alpha \int^* f \, d\mu$ *for* $\alpha \in [0, \infty]$ *and* $f \geq 0$.

Further we mention (cf. (39))

(12) **Lemma.** *If* $A \subset X$ *with* $\mu^*(A) = 0$, *then* $\int^* f \, d\mu = \int^* 1_{X \setminus A} f \, d\mu$ *holds for* $f \geq 0$ *and, in particular,* $\mu^*(A') = \mu^*(A' \setminus A)$ *for* $A' \subset X$.

Proof. This follows from (6) and $\int^* f 1_A \, d\mu \leq \int^* \infty \cdot 1_A \, d\mu = \infty \cdot \mu^*(A) = 0$ by (11) and (3). □

Also it is clear from the definition of the upper integral that

(13) **Lemma.** $f \geq 0$, $\int^* f \, d\mu < \infty \Rightarrow \{f > 0\}$ *is* μ^**–σ–finite.*

Now, σ-finiteness of $\{f > 0\}$ is assumed.

(14) **Proposition.** *The following regular behavior of the upper integral*

$$\int^* f \, d\mu = \sup\{\int^* 1_E f \, d\mu \; : \; E \in \mathcal{B}, \; \mu(E) < \infty\}$$

holds for all $f \geq 0$ *with* $\{f > 0\}$ μ^**–σ–finite.*

Proof. By assumption there are $E_n \in \mathcal{B}$ with $\mu(E_n) < \infty$ and $E_n \uparrow \bigcup_m E_m \supset \{f > 0\}$. Then $1_{E_n} f \uparrow f$. Hence the result follows from (9) and (7). □

On account of (7) and (4) one has

(15) **Corollary.** $\int^* f \, d\mu = \inf\{\int g \, d\mu \; : \; g \geq f, \; g$ *$\mathcal{B}$–measurable*$\}$ *holds for every* $f \geq 0$. *Plainly, the infimum is attained.*

The upper integral could be defined by (15).

Definition. The **completion** of $\mathcal{B}$ with respect to μ is the σ–algebra

(16) $\hat{\mathcal{B}} := \{C \subset X : \mu(B \setminus A) = 0$ *for some* $A, B \in \mathcal{B}, \; A \subset C \subset B\}$.

Plainly, $\mathcal{B} \subset \hat{\mathcal{B}}$. A larger σ–algebra is the **Lebesgue extension** of $\mathcal{B}$ with respect to μ defined by

(17) $\mathcal{B}_L := \{A \subset X : \; A \cap E \in \hat{\mathcal{B}}$ *for every* $E \in \mathcal{B}$ *with* $\mu(E) < \infty\}$.

By (28) below, $\mathcal{B}_L$ is equal to the **Carathéodory extension**

(18) $\mathcal{B}^* := \{A \subset X : \; \mu^*(S) = \mu^*(S \cap A) + \mu^*(S \setminus A)$ *for any* $S \subset X\}$

of $\mathcal{B}$ with respect to μ.

The μ-completion $\hat{\mathcal{B}}$ of $\mathcal{B}$ can be described equivalently as

$$(19) \quad \hat{\mathcal{B}} = \{C \subset X : \textit{there is } B \in \mathcal{B} \textit{ with } \mu^*(C \triangle B) = 0\}.$$

Indeed, for the less trivial inclusion $\supset$ note that $B \setminus D \subset C \subset B \cup D$ where $D \in \mathcal{B}$ is any μ-null set containing $C \triangle B$.

For any measure space (X, B, μ) the relation $B \sim C$ on $\mathcal{B}$ defined by $\mu(B \triangle C) = 0$ is an equivalence relation such that μ is constant on each equivalence class. The set of equivalence classes $[B]$, $B \in \mathcal{B}$, together with the $[0, \infty]$–valued function $[B] \mapsto \mu(B)$, is called the **measure algebra** of $(X, \mathcal{B}, \mu)$. It is partially ordered by $[B] \leq [C]$ if $\mu(B \setminus C) = 0$. By definition, the measure algebra is **Dedekind complete**, if every subset of it has a supremum (see [Hal; §40], [Fre'74; 13B, 41G]), and the measure space $(X, \mathcal{B}, \mu)$ is called **Maharam** or **localizable** if the measure algebra is Dedekind complete and μ is semi-finite (cf. before (47)). It is easy to show that any finite, and hence any σ-finite, measure space is Maharam (cf. the proof of (2.138)). The important fact we want to mention is that the Strong Radon-Nikodým Theorem holds if and only if the measure space is Maharam [Fre'74; 64B].

Recall that for any outer measure ω the set

$$(20) \qquad \mathcal{A} := \{A \subset X : \omega(S) = \omega(S \cap A) + \omega(S \setminus A) \textit{ for any } S \subset X\}$$

is the σ–**algebra of the** ω–**measurable sets** and that by [HewStr; (10.9)]

$$(21) \qquad \omega\Big(\bigcup_n (A_n \cap S)\Big) = \textstyle\sum_n \omega(A_n \cap S)$$

holds for disjoint $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, and $S \subset X$. In particular, $\omega|\mathcal{A}$ is a measure. For $S \subset X$ call $\mathcal{A}_S := \{A \cap S : A \in \mathcal{A}\}$ the **restriction** of $\mathcal{A}$ to S. It is easy to see that $\mathcal{A}_S$ is a σ-algebra on S contained in the σ-algebra $\mathcal{S}$ of $\omega|\mathcal{P}(S)$–measurable sets. Hence $\omega|\mathcal{A}_S$ is a measure on S, extended by $\omega|\mathcal{S}$.

A function $f : X \to [-\infty, \infty]$ is called ω–**integrable** if it is $\omega|\mathcal{A}$–integrable.

In the case of the outer measure μ^*, the σ-algebra $\mathcal{B}$ lies in $\mathcal{B}^*$, and thus $\mu^*|\mathcal{B}^*$ extends μ as a measure. See [Bau; 5.1, 5.3] or [Els; II.4.5]. The measure $\mu^*|\mathcal{B}^*$ is called the **Carathéodory extension of** μ. Obviously $(\mu^*|\mathcal{B}^*)^* \leq \mu^*$. As the infimum is attained in (1) for $\mu^*|\mathcal{B}^*$, also the reverse inequality follows immediately. Therefore

$$(22) \qquad (\mu^*|\mathcal{B}^*)^* = \mu^*$$

holds. The measure $\mu^*|\mathcal{B}^*$ is **complete**, i.e., all subsets of null sets are measurable. Thus $\hat{\mathcal{B}} \subset \mathcal{B}^*$. The measure $\hat{\mu} := \mu^*|\hat{\mathcal{B}}$ is called the **completion** of μ. One easily verifies

(23) $$(\hat{\mu})^* = \mu^*.$$

Recall that the outer measure μ^* has the Fatou property by (10). As mentioned after (10) not all outer measures have that property. However, the Fatou property does not single out those outer measures which are derived from a measure according to (1). More precisely, there is an outer measure ω having the Fatou property such that $(\omega|\mathcal{A})^* \neq \omega$ in contrast to (22), where $\mathcal{A}$ denotes the σ–algebra of ω–measurable sets. As an **example**, let X consist of two points and let $\omega(\emptyset) := 0$, $\omega(X) := 3$, and $\omega(A) := 2$ for $\emptyset \neq A \neq X$, cf. [Zaa; Chap.2, 7.2 Exercise, p. 44, 488]. — Generally an outer measure ω is called **regular**, if $(\omega|\mathcal{A})^* = \omega$ with $\mathcal{A}$ as above. So μ^* is regular by (22). Clearly, an outer measure ω is regular if and only if for every $A \subset X$ there is a $B \in \mathcal{A}$ satisfying $B \supset A$ and $\omega(A) = \omega(B)$.

(24) **Proposition.** *Regular outer measures have the Fatou property.*

Proof. Let (A_n) be an increasing sequence of subsets of X. By the regularity of the outer measure ω there are $B_n \in \mathcal{A}$ with $B_n \supset A_n$ and $\omega(A_n) = \omega(B_n)$. Then $\omega(\bigcup_n A_n) \leq \omega(\bigcup_n \bigcap_{m \geq n} B_m) = \lim_n \omega(\bigcap_{m\geq n} B_m) \leq \liminf_n \omega(B_n) = \liminf_n \omega(A_n) = \lim_n \omega(A_n)$, where the two inequalities and the last equality hold because of the monotonicity of ω and the first equality holds because of the continuity from below of the measure $\omega|\mathcal{A}$. The reverse inequality $\lim_n \omega(A_n) \leq \omega(\bigcup_n A_n)$ is obvious. □

(25) **Proposition.** *For all $A \subset X$ one has*

$$\mu^*(A) = 0 \quad \Leftrightarrow \quad A \textit{ is a subset of some } \mu\textit{–null set} \quad \Rightarrow \quad A \in \hat{\mathfrak{B}}.$$

Proof. This follows from (1) and (16). □

(26) **Proposition.** *For all $f \geq 0$ one has*

$$\int^* f\,d\mu = 0 \quad \Leftrightarrow \quad \mu^*(\{f > 0\}) = 0 \quad \Rightarrow \quad f \textit{ is } \hat{\mathfrak{B}}\textit{–measurable.}$$

Proof. The implication $\Leftarrow$ is clear by (12). Now let $\int^* f\,d\mu = 0$. Then $\mu^*(\{f > \alpha\}) \leq \int^* \frac{1}{\alpha} f\,d\mu = 0$ for all $\alpha > 0$ by (3), (7), and (11). Therefore $\mu^*(\{f > 0\}) = 0$ by (10). This implies $\{f > \alpha\} \in \hat{\mathfrak{B}}$ by (25) for all $\alpha \geq 0$. Hence f is $\hat{\mathfrak{B}}$–measurable. □

(27) **Proposition.** *The Carathéodory extension* $\mathcal{B}^*$ *satisfies*

$$\mathcal{B}^* = \{A \subset X \,:\, \mu(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \text{ for all } B \in \mathcal{B}\}.$$

Proof. $\mathcal{B}^*$ is contained in the right hand side of (27), which we call $\tilde{\mathcal{B}}$. Now let $A \in \tilde{\mathcal{B}}$ and $S \subset X$. Because of the subadditivity of μ^* it suffices to show that $\mu^*(S) \geq \mu^*(S \cap A) + \mu^*(S \setminus A)$. By (1) there is $B \in \mathcal{B}$ with $B \supset S$ and $\mu(B) = \mu^*(S)$. Then $\mu^*(S) = \mu(B) = \mu^*(B \cap A) + \mu^*(B \setminus A) \geq \mu^*(S \cap A) + \mu^*(S \setminus A)$ since $A \in \tilde{\mathcal{B}}$ and since μ^* is monotone. □

This shows that $\mathcal{B}^*$ is the largest of all σ–algebras $\mathcal{C}$ on X such that $\mathcal{C} \supset \mathcal{B}$ and $\mu^*|\mathcal{C}$ is a measure. We now show

(28) **Proposition.** *The algebras* $\mathcal{B}^*$ *and* $\mathcal{B}_L$ *coincide.*

Proof. As to $\mathcal{B}^* \subset \mathcal{B}_L$, let $A \in \mathcal{B}^*$ and consider $E \in \mathcal{B}$ with $\mu(E) < \infty$. There exists $F \in \mathcal{B}$ with $F \supset A \cap E$ and $\mu(F) = \mu^*(A \cap E)$. As $A \cap E \in \mathcal{B}^*$ one gets $\mu(F) = \mu^*(A \cap E) + \mu^*(F \setminus (A \cap E))$. Therefore $\mu^*(F \setminus (A \cap E)) = 0$ and $A \cap E \in \hat{\mathcal{B}}$ follows by (25).

To show the reverse inclusion using (27), let $A \in \mathcal{B}_L$ and $B \in \mathcal{B}$ with $\mu(B) < \infty$. Since $A \in \mathcal{B}_L$ one has $B \cap A \in \hat{\mathcal{B}}$ by (17). So there is $F \in \mathcal{B}$ with $B \supset F \supset B \cap A$, $\mu^*(F \setminus (B \cap A)) = 0$. Then the outer measure μ^* satisfies $\mu^*(B \cap A) + \mu^*(B \setminus A) \leq \mu(F) + \mu(B \setminus F) + \mu^*(F \setminus (B \cap A)) = \mu(B)$. □

Consequences of (28), the fact that $\hat{\mathcal{B}} \subset \mathcal{B}^*$, and (25) are

(29) **Proposition.** *For all* $A \subset X$ *one has*

$$A \in \mathcal{B}^* \;\Leftrightarrow\; A \cap E \in \mathcal{B}^* \text{ for all } E \in \mathcal{B} \text{ with } \mu(E) < \infty.$$

(30) **Proposition.** *For all* $A \subset X$ *one has*

$$\begin{aligned} & A \in \mathcal{B}^* \text{ with } \mu^*(A) < \infty \\ \Leftrightarrow\;\; & \mu^*(B \setminus A) = 0 \text{ for some } B \in \mathcal{B} \text{ with } B \supset A \text{ and } \mu(B) < \infty \\ \Leftrightarrow\;\; & \mu^*(A \setminus B) = 0 \text{ for some } B \in \mathcal{B} \text{ with } B \subset A \text{ and } \mu(B) < \infty \\ \Rightarrow\;\; & A \in \hat{\mathcal{B}}. \end{aligned}$$

A generalisation of a part of (30) is

(31) **Lemma.** *Let $f \geq 0$ be $\mathcal{B}^*$–measurable with $\{f > 0\}$ μ^*–σ–finite. Then f is $\hat{\mathcal{B}}$–measurable.*

Proof. Use $\mathcal{B}^*$–step functions $s_n \uparrow f$. By (30) these are $\hat{\mathcal{B}}$–measurable. □

(32) **Definition.** A set $A \subset X$ is called a **local μ–null set**, if $\mu^*(A \cap E) = 0$ for all $E \in \mathcal{B}$ with $\mu(E) < \infty$.

Obviously, the union of countably many local μ–null sets is a local μ–null set. Also every μ–null set, and more generally every μ^*–null set is a local μ–null set, and

(33) **Lemma.** *$A \subset X$ is a local null set if and only if $\mu^*(S) \in \{0, \infty\}$ for all $S \subset A$.*

Furthermore by (29) and (25) one has

(34) **Proposition.** *Every local null set is μ^*–measurable.*

If μ is σ–finite then $\mathcal{B}^* = \hat{\mathcal{B}}$ and every local μ–null set is a $\hat{\mu}$–null set. This follows from (30) and (32). But in general $B^* \neq \hat{B}$, see (5.1) (ζ), and there are local null sets in $\mathcal{B}$ which are not null sets, see (5.1) (γ). For $\hat{\mathcal{B}} \neq \mathcal{B}^*$ consider also the following example where $\mathcal{B}^*$ is even larger than the σ–algebra generated by $\mathcal{B}$ and the ideal of the local null sets.

(35) **Example.** Let X be an uncountable set and μ the counting measure on $\mathcal{B} := \{A \subset X : A \text{ or } X \setminus A \text{ is countable}\}$. Then $\hat{\mathcal{B}} = \mathcal{B}$ holds, $\emptyset$ is the only local null set, $\mathcal{B}^* = \mathcal{P}(X)$, and μ^* is the counting measure on $\mathcal{P}(X)$. □

Proposition. *The integral with respect to $\mu^*|\mathcal{B}^*$ satisfies*

(36) $$\int^* f \, d\mu^*|\mathcal{B}^* = \int^* f \, d\mu \quad \textit{for all } f \geq 0 \textit{ (cf. (22))},$$

(37) $$\int f \, d\mu^*|\mathcal{B}^* = \int^* f \, d\mu \quad \textit{for all } \mathcal{B}^*\textit{–measurable } f \geq 0 \textit{ (cf. (4))}.$$

Proof. Since every set $A \in \mathcal{B}^*$ is contained in a set $B \in \mathcal{B}$ with $\mu^*(A) = \mu(B)$ the step functions s in (2) may be replaced by $\mathcal{B}^*$–step functions and $\int s \, d\mu$ by $\int s \, d\mu^*|\mathcal{B}^*$. This proves (36) and shows that (37) holds for $\mathcal{B}^*$–step functions f. The general case now follows by representing f as the limit of an increasing sequence of $\mathcal{B}^*$–step functions and using the Fatou property of the integral and the upper integral. □

Now we are able to show the following **restricted additivity** of $\int^*$.

(38) **Proposition.** *For $g \geq 0$ $\mathcal{B}^*$–measurable and $f \geq 0$ one has*

$$\int^* f\,d\mu = \int^* \inf\{f,g\}\,d\mu + \int^* \sup\{0, f-g\}\,d\mu,$$

agreeing that $\infty - \infty := 0$, $\infty - \alpha := \infty$, $\alpha - \infty := -\infty$ for $\alpha \in \mathbb{R}$.

Proof. Let $i := \inf\{f,g\}$, $j := \sup\{0, f-g\}$ and consider a $\mathcal{B}^*$–measurable h with $h \geq f$. Then $i' := \inf\{h,g\}$ and $j' := \sup\{0, h-g\}$ are $\mathcal{B}^*$–measurable and satisfy $i' \geq i$, $j' \geq j$. By (7) and (37) one has $\int^* i\,d\mu + \int^* j\,d\mu \leq \int^* i'\,d\mu + \int^* j'\,d\mu = \int i'\,d\mu^*|\mathcal{B}^* + \int j'\,d\mu^*|\mathcal{B}^* = \int (i'+j')\,d\mu^*|\mathcal{B}^* = \int h\,d\mu^*|\mathcal{B}^*$. Hence (15) and (36) yield $\int^* i\,d\mu + \int^* j\,d\mu \leq \int^* f\,d\mu^*|\mathcal{B}^* = \int^* f\,d\mu$. The reverse relation follows from subadditivity (8). □

Applying (38) with $g := \infty \cdot 1_A$ generalizes (12) because of (26).

(39) **Corollary.** $\int^* f\,d\mu = \int^* 1_A f\,d\mu + \int^* 1_{X\setminus A} f\,d\mu$ *for $f \geq 0$, $A \in \mathcal{B}^*$.*

In what follows the question is treated whether the upper integral can be approximated by integrals from below.

(40) **Proposition.** *For $f \geq 0$ let $a := \int^* f\,d\mu$ and set*

$$\begin{aligned}
b &:= \sup\{\textstyle\int s\,d\mu^*|\mathcal{B}^* : s\ \mathcal{B}^*\text{–step function},\ 0 \leq s \leq f\},\\
c &:= \sup\{\textstyle\int s\,d\mu : s\ \text{step function},\ 0 \leq s \leq f\},\\
b' &:= \sup\{\textstyle\int g\,d\mu^*|\mathcal{B}^* : g\ \mathcal{B}^*\text{–measurable},\ 0 \leq g \leq f\},\\
c' &:= \sup\{\textstyle\int g\,d\mu : g\ \text{measurable},\ 0 \leq g \leq f\}.
\end{aligned}$$

Then the following statements hold:

(i) $b = b'$, $c = c'$ *and the suprema in b' and c' are attained.*
(ii) $a \geq b \geq c$.
(iii) f $\mathcal{B}^*$*–measurable* $\Rightarrow a = b$.
(iv) f $\hat{\mathcal{B}}$*–measurable* $\Rightarrow a = b = c$.
(v) $b < \infty \Rightarrow b = c$.
(vi) $a < \infty$, $a = b \Rightarrow f$ $\hat{\mathcal{B}}$*–measurable.*

Proof. **(i)** is elementary. **(ii)** $b \geq c$ is obvious. $a \geq b$ follows from (36) and by applying (4), (7) to $\mu^*|\mathcal{B}^*$. **(iii)** holds by (37) because of the Fatou property. **(iv)** Using (37), $a = \int f\,d\hat{\mu}$ follows. By the Fatou property the latter integral is the supremum of integrals of $\hat{\mathcal{B}}$–step functions $\leq f$. Since these functions dominate step functions with same integrals, $a = c$ follows.