A FIRST COURSE IN DIFFERENTIAL GEOMETRY

Chuan-Chih Hsiung

A First Course in Differential Geometry

CHUAN-CHIH HSIUNG

Lehigh University

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Preface

According to a definition stated by Felix Klein in 1872, we can use geometric transformation groups to classify geometry. The study of properties of geometric figures (curves, surfaces, etc.) that are invariant under a given geometric transformation group G is called the geometry belonging to G. For instance, if G is the projective, affine, or Euclidean group, we have the corresponding projective, affine, or Euclidean geometry.

The differential geometry of a geometric figure F belonging to a group G is the study of the invariant properties of F under G in a neighborhood of an element of F. In particular, the differential geometry of a curve is concerned with the invariant properties of the curve in a neighborhood of one of its points. In analytic geometry the tangent of a curve at a point is customarily defined to be the limit of the secant through this point and a neighboring point on the curve, as the second point approaches the first along the curve. This definition illustrates the nature of differential geometry in that it requires a knowledge of the curve only in a neighborhood of the point and involves a limiting process (a property of this kind is said to be local). These features of differential geometry show why it uses the differential calculus so extensively. On the other hand, local properties of geometric figures may be contrasted with global properties, which require knowledge of entire figures.

The origins of differential geometry go back to the early days of the differential calculus, when one of the fundamental problems was the determination of the tangent to a curve. With the development of the calculus, additional geometric applications were obtained. The principal contributors in this early period were Leonhard Euler (1707–1783), Gaspard Monge (1746–1818), Joseph Louis Lagrange (1736–1813), and Augustin Cauchy (1789–1857). A decisive step forward was taken by Karl Friedrich Gauss (1777–1855) with his development of the intrinsic geometry on a surface. This idea of Gauss was generalized to $n(>3)$-dimensional space by Bernhard Riemann (1826–1866), thus giving rise to the geometry that bears his name.

This book is designed to introduce differential geometry to beginning graduate students as well as advanced undergraduate students (this introduction in the latter case is important for remedying the weakness of geometry in the usual undergraduate curriculum). In the last couple of decades differential geometry, along with other branches of mathematics, has been highly developed. In this book we will study only the traditional topics, namely, curves and surfaces in a three-dimensional Euclidean space E^3. Unlike most classical books on the subject, however, more attention is paid here to the relationships between local and global properties, as opposed to local properties only. Although we restrict our attention to curves and surfaces in E^3, most global theorems for curves and surfaces in this book can be extended to either higher dimensional spaces or more general curves and surfaces or both. Moreover, geometric interpretations are given along with analytic expressions. This will enable students to make use of geometric intuition, which is a precious tool for studying geometry and related problems; such a tool is seldom encountered in other branches of mathematics.

We use vector analysis and exterior differential calculus. Except for some tensor conventions to produce simplifications, we do not employ tensor calculus, since there is no benefit in its use for our study in space E^3. There are four chapters whose contents are, briefly, as follows.

Chapter 1 contains, for the purpose of review and for later use, a collection of fundamental material taken from point-set topology, advanced calculus, and linear algebra. In keeping with this aim, all proofs of theorems are self-contained and all theorems are expressed in a form suitable for direct later application. Probably most students are familiar with this material except for Section 6 on differential forms.

In Chapter 2 we first establish a general local theory of curves in E^3, then give global theorems separately for plane and space curves, since those for plane curves are not special cases of those for space curves. We also prove one of the fundamental theorems in the local theory, the uniqueness theorem for curves in E^3. A proof of this existence theorem is given in Appendix 1.

Chapter 3 is devoted to a local theory of surfaces in E^3. For this theory we only state the fundamental theorem (Theorem 7.3), leaving the proofs of the uniqueness and existence parts of the theorem to, respectively, Chapter 4 (Section 4) and Appendix 2.

Chapter 4 begins with a discussion of orientation of surfaces and surfaces of constant Gaussian curvature, and presents various global theorems for surfaces.

Most sections end with a carefully selected set of exercises, some of which supplement the text of the section; answers are given at the end of the book. To allow the student to work independently of the hints that accompany some of the exercises, each of these is starred and the hint together with the answer appear at the end of the book. Numbers in brackets refer to the items listed in the Bibliography at the end of the book.

Two enumeration systems are used to subdivide sections; in Chapters 1 (except Sections 4 and 7) and 2, triple numbers refer to an item (e.g., a theorem or definition), whereas in Chapters 3 and 4 such an item is referred to by a double number. However, there should be no difficulty in using the book for reference purposes, since the title of the item is always written out (e.g., Corollary 5.1.6 of Chapter 1 or Lemma 1.5 of Chapter 3).

This book can be used for a full-year course if most sections of Chapter 1 are studied thoroughly.

For a one-semester course I suggest the use of the following sections:

Chapter 1: Sections 3.1, 3.2, 3.3, 6.

Chapter 2: Section 1.1 (omit 1.1.4–1.1.6), Section 1.2 (omit 1.2.6, 1.2.7), Section 1.3 (omit 1.3.7–1.3.12), Sections 1.4 and 1.5 (omit 1.5.5); Section 2 (omit 2.3, 2.5, 2.6.4–2.6.6, 2.9–2.11, 2.14–2.23); Section 3 (omit 3.1.8–3.1.14).

Chapter 3: Section 1 (omit the proof of 1.6, 1.7, 1.8, the proof of 1.10, 1.11–1.13, 1.15–1.18); Section 2 (omit the proof of 2.4); Sections 3–9; Section 10 (omit the material after 10.7).

Chapter 4: Section 1 (omit the proofs of 1.3 and 1.4); Section 3 (omit 3.14); Sections 4 and 5.

For a course lasting one quarter I suggest omission of the following material from the one-semester outline above: Chapter 2: the second proof of 2.6, 3.2; Chapter 3: the details of 1.3 and 1.4, the proof of 5.7, Section 6, the proofs of 8.1 and 8.2; Chapter 4: Section 5.

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CHUAN-CHIH HSIUNG

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General Notation and Definitions

NOTATION

Symbol	*Usage*	*Meaning*
$\in$	$x \in A$	x is an element of the set A
$\notin$	$x \notin A$	x is not an element of the set A
$\subset$	$B \subset A$	The set B is a subset of the set A
$\varnothing$	$\varnothing$	The empty set
$\cap$	$A \cap B$ $\cap A_i$	Intersection of the sets A and B Intersection of all the sets A_i
$\cup$	$A \cup B$ $\cup A_i$	Union of the sets A and B Union of all the sets A_i
$\{\ \}$	$\{x \mid \cdots\}$	The set of all x such that $\cdots$
$\Rightarrow$	$\cdots \Rightarrow$———	$\cdots$ implies———
$\Leftrightarrow$	$\cdots \Leftrightarrow$———	$\cdots$ if and only if———
$\to$	$A \to B$	Function on the set A to the set B
$\mapsto$	$x \mapsto x^2$	Function assigning x^2 to x
$[,]$	$[a, b]$	$\{x \mid a \leqslant x \leqslant b\}$
$(,)$	(a, b)	$\{x \mid a < x < b\}$

DEFINITIONS

A *function* f on a set A to a set B is a rule that assigns to each element x of A a unique element $f(x)$ of B. The element $f(x)$ is called the *value* of f at x, or the *image* of x under f. The set A is called the *domain* of f, the set B is often called the *range* of f, and the subset of B, denoted by $f(A)$, consisting of all elements of the form $f(x)$ is called the *image* of f.

If both f_1 and f_2 are functions on A to B, then $f_1 = f_2$ means that $f_1(x) = f_2(x)$ for all $x \in A$.

Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Then the function $g(f)\colon A \to C$, whose value on each $x \in A$ is the element $g(f(x)) \in C$, is called the *composite function* of f and g, denoted by $g \circ f$.

If $f\colon A \to B$ is a function, C is a subset of A, and D is a subset of B, the *restriction* of f to C is the function $f|C\colon C \to B$ defined by the same rule as f, but applied only to elements of C, and the subset of A consisting of all $x \in A$ such that $f(x) \in D$ is called the *inverse image* of D and is denoted by $f^{-1}(D)$.

A function $f\colon A \to B$ is said to be *one-to-one* or *injective* if $x \neq y$ implies $f(x) \neq f(y)$. An injective function is called an *injection*. f is said to be *onto* or *surjective* if to each element $b \in B$ there exists at least one element $a \in A$ such that $f(a) = b$. A surjective function is called a *surjection*. A function that is both injective and surjective is said to be *bijective*. A bijective function is also called a *bijection*.

Note that under a bijective function $f\colon A \to B$, each element $b \in B$ is the image of one and only one element $a \in A$. We then have an inverse function f^{-1}, defined throughout B, which assigns to each element $b \in B$ the unique element $a \in A$ such that $b = f(a)$.

Let k be a nonnegative integer. A function on a Euclidean n-space E^n to the real line E^1 is said to be *of class* C^k (respectively, C^∞) or a C^k (respectively, C^∞) *function* if its partial derivatives of orders up to and including k (respectively, of all orders) exist and are continuous. A C^0 function means merely a continuous function.

The words "set," "space," and "collection" are synonymous, as are the words "function" and "mapping."

Contents

1

Euclidean Spaces

This chapter contains, for a Euclidean space of three dimensions (extension to higher dimensions is virtually automatic), the fundamental material that is necessary for later developments of this book. Although most students are probably familiar with a great part of the material, its placement in one chapter makes it convenient for purposes of review and also allows us to bring out more clearly relationships among certain notions. Depending on the backgrounds of the students, certain sections may be selected for more thorough study.

1. POINT SETS

1.1. Neighborhoods and Topologies. Let E^3 be a Euclidean three-dimensional space. In the usual sense, in E^3 we take a fixed right-handed rectangular trihedron $\mathbf{0x}_1\mathbf{x}_2\mathbf{x}_3$ (see Fig. 1.1), that is, a point $\mathbf{0}$, called the origin of E^3, and mutually orthogonal coordinate axes $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, whose positive directions form a right-handed trihedron. Then relative to $\mathbf{0x}_1\mathbf{x}_2\mathbf{x}_3$ a point $\mathbf{x}$ in E^3 has coordinates (x_1, x_2, x_3). More generally, we have the following definition.

*1.1.1. **Definition.*** A *Euclidean n*-dimensional space E^n is the set of all ordered n-tuples $\mathbf{x}=(x_1,\cdots,x_n)$ of real numbers. Such an n-tuple is a ***point*** in E^n.

In accordance with our stated purpose, here and throughout this book we limit our discussions to $n=1,2,3$.

Let $u_1,\cdots,u_n$ be real-valued functions on E^n such that for each point $\mathbf{x}=(x_1,\cdots,x_n)$,

$$u_1(\mathbf{x})=x_1,\cdots,u_n(\mathbf{x})=x_n.$$

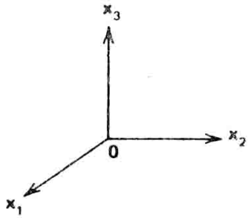

Figure 1.1

These functions $u_1, \cdots, u_n$ are called the *natural coordinate functions* of E^n.

The distance $d(\mathbf{x},\mathbf{y})$ between two points $\mathbf{x}=(x_1,\cdots,x_n)$ and $\mathbf{y}=(y_1,\cdots,y_n)$ in E^n is defined by the formula

$$d^2(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} (x_i - y_i)^2, \qquad d(\mathbf{x},\mathbf{y}) \geqslant 0. \tag{1.1.1}$$

It is obvious that $d(\mathbf{x},\mathbf{y})=0$ if and only if $x_i = y_i$, $i=1,\cdots,n$, that is, if and only if $\mathbf{x}$ coincides with $\mathbf{y}$. Furthermore, we have $d(\mathbf{x},\mathbf{y})=d(\mathbf{y},\mathbf{x})$ and the triangle inequality for $\mathbf{z}\in E^n$

$$d(\mathbf{x},\mathbf{y}) + d(\mathbf{y},\mathbf{z}) \geqslant d(\mathbf{x},\mathbf{z}). \tag{1.1.2}$$

1.1.2. ***Definition.*** An *open spherical neighborhood* of a point $\mathbf{p}_0$ in E^n is the set of the form

$$\{\mathbf{p}\in E^n \mid d(\mathbf{p},\mathbf{p}_0) < \rho\}, \tag{1.1.3}$$

where $\rho > 0$. More generally, a *neighborhood* of $\mathbf{p}_0$ is any set that contains a spherical neighborhood of $\mathbf{p}_0$.

For $n=3$ it is convenient to use open spherical neighborhoods. However, for $n=2$ a neighborhood of $\mathbf{p}_0$ is any set that contains some open disk $\{p\in E^2 \mid d(\mathbf{p},\mathbf{p}_0)<\rho\}$ about $\mathbf{p}_0$, and for $n=1$ it is an open interval containing $\mathbf{p}_0$.

We can easily obtain Lemma 1.1.3.

1.1.3. ***Lemma.*** *The neighborhoods of a point $\mathbf{p}_0$ in E^n have the following properties:*

(a) *$\mathbf{p}_0$ belongs to any neighborhood of $\mathbf{p}_0$.*

(b) *If U is a neighborhood of $\mathbf{p}_0$, and V a set such that $V \supset U$, then V is also a neighborhood of $\mathbf{p}_0$.*

(c) *If U and V are neighborhoods of $\mathbf{p}_0$, so is $U \cap V$.*

(d) *If U is a neighborhood of $\mathbf{p}_0$, there is a neighborhood V of $\mathbf{p}_0$ such that $V \subset U$ and V is a neighborhood of each of its points.*

1.1.4. Definition. In general, *a topological space* is a set S together with an assignment to each element $p_0 \in S$ of a collection of open subsets of S (see Definition 1.2.1), to be called *neighborhoods* of p_0, satisfying the four properties listed in Lemma 1.1.3, and the collection of the neighborhoods of all points of S is a *topology* for the space S.

Thus a Euclidean space E^n and the unit sphere in E^3 with center at the origin are both topological spaces.

1.1.5. Definition. Let S be a topological space, T a subset of S, and p a point of T. Then a subset U of T is a *neighborhood* of p in T if $U = T \cap V$, where V is a neighborhood of p in S. The neighborhoods U in T so defined have the four properties in Lemma 1.1.3. When T is made into a topological space by defining neighborhoods in this way, it is a *subspace* of S, and all the neighborhoods form a *relative topology* of S for the space T.

In the remainder of this section, unless stated otherwise, all spaces are supposed to be topological and all sets are to be in a general topological space, although we shall be interested only in spaces E^n for $n = 1,2,3$.

1.1.6. Definition. With respect to a subset T of a space S, each point p has one of the following three properties:

(a) p is *interior* to T if $p \in T$ and T is a neighborhood of p. The set of all the points interior to T is the *interior* of T.
(b) p is *exterior* to T if $p \notin T$ and there is a neighborhood of p that is disjoint from T, (i.e., has no points in common with T).
(c) p is a *boundary point* of T if p is neither interior nor exterior to T. The set of all boundary points of T is the *boundary* of T and is denoted by ∂T.

From the definition above it follows that an interior point of T is surrounded completely by points of T, that there are no points of T that are arbitrarily close to an exterior point of T, and that a boundary point of T may or may not belong to T.

The following is a frequently used method of obtaining new spaces from given spaces.

Let S and T be nonempty spaces. The set $S \times T$, called the *Cartesian product* of S and T, is defined to be the set of all ordered pairs (p, q) where $p \in S$ and $q \in T$. This set is made into a space as follows. If $(p, q) \in S \times T$, then a neighborhood of (p, q) is any set containing a set of the form $U \times V$, where U is a neighborhood of p in S, and V is a neighborhood of q in T. It is not hard to see that the neighborhood axioms a–d of Lemma 1.1.3 are satisfied.

*1.1.7. **Definition.*** $S \times T$, made into a space as just described, is the *topological product* of S and T.

Examples. 1. If $S = T = E^1$, then $S \times T$ is the plane with its usual topology (i.e., is E^2).

2. If $S = E^2$ and $T = E^1$, then $S \times T = E^3$. In general, $E^m \times E^n = E^{m+n}$.
3. If S is an interval on E^1, and T is a circle, then $S \times T$ is a cylinder.
4. The torus is the topological product of a circle with itself.

Exercises

1. Prove Lemma 1.1.3.
2. Let $S = \{\text{all } (x, y) \text{ with } x \text{ and } y \text{ rational numbers}\}$. (a) What is the interior of S? (b) What is the boundary of S?

1.2. Open and Closed Sets, and Continuous Mappings

*1.2.1. **Definition.*** A subset T of a space S is *open* if every point of T is interior to T; this is the same as saying that no boundary point of T belongs to T. A subset T of a space S is *closed* if every point of S that is not in T is in fact exterior to T; this is the same as saying that every boundary point of T is in T. The empty set, denoted by $\varnothing$, is an open set that contains no elements and is therefore a subset of every set.

The behavior of open and closed sets under the operations of union and intersection is of fundamental importance and is described by the following theorem.

*1.2.2. **Theorem.*** (a) *The union of any collection of open sets in a space S is open.*

(b) *The intersection of a finite collection of open sets in S is open.*
(c) *The intersection of any collection of closed sets in S is closed.*
(d) *The union of a finite collection of closed sets in S is closed.*

Proof. (a) Let $\{U_i\}$ be a collection of open sets in S, where i ranges over some set of indices. Let $U = \cup U_i$ and take p in U. Then $p \in U_i$ for some i, and by Definition 1.2.1, U_i is a neighborhood of p. Since $U \supset U_i$, U is a neighborhood of p by Lemma 1.1.3(b). Thus U is a neighborhood of each of its points and is therefore open by Definition 1.2.1.

(b) Let U_1 and U_2 be open sets in S and take $p \in U_1 \cap U_2$. Then by Definition 1.2.1, U_1 and U_2 are neighborhoods of p. From Lemma 1.1.3(c) it thus follows that $U_1 \cap U_2$ is a neighborhood of p. Since p is arbitrary, $U_1 \cap U_2$ is open by Definition 1.2.1. Using mathematical induction, we can extend this to a finite collection of open sets in S.

Parts (c) and (d) of Theorem 1.2.2 are obtained from parts (a) and (b) by considering the complementary sets.

Remark. The statement of Theorem 1.2.2b may not be true if "finite" is replaced by "infinite." For example, if S is the real line E^1, and U_n is the open interval $(-1/n, 1/n)$, then each U_n is an open set, but the intersection of all the U_n is the point 0, which is not open.

1.2.3. Definition. A point p is a *limit* (or *cluster* or *accumulation*) *point* of a set T if every neighborhood of p contains a point of T distinct from p. The *closure* of a subset T of S, denoted by $\bar{T}$, is the union of T and the set of its limit points.

Example. Consider the set

$$S=\{p\in E^2 \mid 0<d(\mathbf{p},\mathbf{0})<1\} \cup \{\text{the point } (0,2)\},$$

where $\mathbf{0}$ is the point (0,0). The boundary of S consists of the circumference, where $d(\mathbf{p},\mathbf{0})=1$, and the two points $\mathbf{0}$ and (0,2). The interior of S is the set of points $\mathbf{p}$ with $0<d(\mathbf{p},\mathbf{0})<1$, and the closure of S is the set consisting of the point (0,2), together with the unit disk, the set of all points $\mathbf{p}$ such that $d(\mathbf{p},\mathbf{0})\leq 1$.

1.2.4. Definition. Let T be a subset of a space S. The set of all points of S that are not in T is the *complement* of T in S, and is denoted by $S-T$.

The following lemma is an immediate consequence of Definitions 1.2.1 and 1.2.4.

1.2.5. Lemma. *A subset T of a space S is open* (*respectively, closed*) *if and only if $S-T$ is closed* (*respectively, open*).

1.2.6. Theorem. *Let T be any subset of a space S. Then T is closed if and only if $T=\bar{T}$.*

Proof. First suppose that T is closed. Then $S-T$ is open. If $S-T$ contains a limit point p of T, then every neighborhood $N(p)$ of p contains a point of T, and $N(p)\not\subset S-T$. This contradicts the fact that $S-T$ is open. Hence $T=\bar{T}$.

Conversely, suppose that $T=\bar{T}$. Let $p\in S-T$. Then $p\notin \bar{T}$, and p is not a limit point of T. Thus p has a neighborhood that contains no point of T

and belongs to $S-T$. This shows that $S-T$ is open, and therefore that T is closed.

1.2.7. Definition. Let S and T be two spaces. A mapping $f\colon S\to T$ is *continuous at a point* $p\in S$ if for any neighborhood V of $f(p)$ in T there is a neighborhood U of p in S such that $f(U)\subset V$. f is continuous if it is *continuous* at each point p in S.

1.2.8. Definition. Let S and T be two spaces, and $f\colon S\to T$ be a bijection. If both f and its inverse f^{-1} are continuous, then f is a *homeomorphism*, and S and T are *homeomorphic*.

Exercises

1. By constructing an example, show that the union of an infinite collection of closed sets may not be closed.

*2. Show that (a) the interior of a set S is the largest (in the sense of inclusion) open set contained in S, and (b) the closure of a set S is the smallest closed set that contains S.

*3. Let C be a closed set, and U an open set. Prove: (a) $C-U=\{\text{all } p\in C \text{ with } p\notin U\}$ is closed, and (b) $U-C$ is open.

4. Show that if p is a limit point of a set S in E^n, then every neighborhood of p contains infinitely many points of S.

5. If S and T both are the real line, the usual definition of continuity is that $f\colon S\to T$ is continuous at $x\in S$ if for any $\varepsilon>0$ there is a $\delta>0$ such that $|f(x')-f(x)|<\varepsilon$ whenever $x'\in S$ and $|x'-x|<\delta$. Prove that this is equivalent to Definition 1.2.7.

6. Let S and T be two spaces and $f\colon S\to T$ a mapping. Prove that f is continuous if and only if the inverse image of any open set in T is open in S. Use this to show that the composition of continuous mappings is continuous.

7. Let S be the union of two closed sets U and V, and $f\colon S\to T$ a mapping. Suppose that the restriction of f to U and to V are continuous mappings of U and V into T, respectively. Show that f is continuous. Given an example to show that this does not hold if U and V are not closed.

*Asterisks indicate exercises for which hints are provided at the end of the book.