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Preface

These notes are an elaboration of the first part of a course on foliations which I have given at Strasbourg in 1976 and at Tunis in 1977.

They are concerned mostly with dynamical systems in dimensions one and two, in particular with a view to their applications to foliated manifolds. An important chapter, however, is missing, which would have been dealing with structural stability.

The publication of the French edition was realized by the efforts of the secretariat and the printing office of the Department of Mathematics of Strasbourg. I am deeply grateful to all those who contributed, in particular to Mme. Lambert for typing the manuscript, and to Messrs. Bodo and Christ for its reproduction.

Strasbourg, January 1979.

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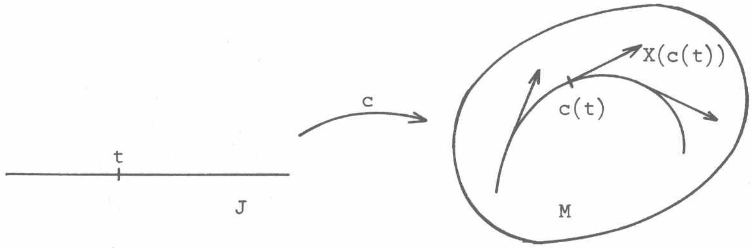
Chapter I. Vector Fields on Manifolds

1. INTEGRATION OF VECTOR FIELDS

Let M be a differentiable manifold without boundary of dimension m and of class C^s, $2 \leqslant s \leqslant +\infty$ (respectively analytic), and let X be a vector field on M of class C^r, $1 \leqslant r \leqslant s-1$ (respectively analytic).

1.1. DEFINITION. An <u>integral curve</u> of X is a map c of class C^1 of an interval J of $\mathbb{R}$ into M satisfying $c'(t) = X(c(t))$ for all $t \in J$.



1.2. <u>Examples</u>.

i) If y is a zero of X then the constant map of $\mathbb{R}$ onto y is an integral curve of X.

In this case y is called a <u>singular point</u> of X. A point of M where X does not vanish is a <u>regular point</u>.

ii) If c is an integral curve of X, so is the map $t \longmapsto c(t+\tau)$, for all $\tau \in \mathbb{R}$.

iii) If c is an integral curve of X the map $t \mapsto c(-t)$ is an integral curve of the field $-X$.

iv) Let $q: \tilde{M} \to M$ be a covering map. Then the tangent bundle $T(\tilde{M})$ is isomorphic to the inverse image $q^*T(M)$ of the tangent bundle $T(M)$ by the projection q. Hence there is a uniquely defined vector field $\tilde{X}$ on $\tilde{M}$ (in the same differentiability class as X) such that $q^T \circ \tilde{X} = X \circ q$.

In this case the projection $c = q \circ \tilde{c}$ of an integral curve $\tilde{c}$ of $\tilde{X}$ is an integral curve of X.

Conversely every integral curve of X is the projection of an integral curve of $\tilde{X}$.

v) Let X and Y be vector fields on manifolds M and N respectively and $h: M \to N$ a differentiable map satisfying $h^T \circ X = Y \circ h$. Then the image under h of any integral curve of X is an integral curve of Y.

1.3. <u>Remarks</u>.

i) Let $(y_1, \ldots, y_m)$ be a local system of coordinates on an open set U of M, and let X be expressed on U as $\sum_i a_i \frac{\partial}{\partial y_i}$. Then the integral curves of X in U are the <u>solutions of the (autonomous) system of differential equations</u> $y_i' = a_i(y_1, \ldots, y_m)$, $i = 1, \ldots, m$.

ii) If $N = M \times \mathbb{R}$, a vector field Y on N of the form $Z(y,s) + \frac{\partial}{\partial s}$, with $Z(y,s) \in T_yM$, corresponds locally to <u>a non-autonomous system of differential equations</u> $y_i' = b_i(y_1, \ldots, y_m, t)$, $i = 1, \ldots, m$.

The above remark (i) allows a reformulation of the local existence and uniqueness theorems for solutions of differential equations, as well as of the statements about their differentiable dependence on initial conditions as follows (cf. [10], [12]):

1.4. <u>THEOREM</u>. For every point y of M and for every real number τ there exist an open neighbourhood U of y in M,

a number $\varepsilon > 0$,

a map Φ of class C^r (respectively analytic) of $(\tau-\varepsilon, \tau+\varepsilon)\times U$ into M,

satisfying for every point x of U the following properties:

a) $t \to \Phi(t,x)$ is an integral curve of X;

b) $\Phi(\tau, x) = x$;

c) if c is an integral curve of X defined on an interval containing τ and such that $c(\tau) = x$, then $c(t) = \Phi(t,x)$ in a neighbourhood of τ.

Consequences:

i) Two integral curves of X intersecting in a point coincide in a neighbourhood of this point;

ii) let U_i, ε_i, Φ_i, $i = 1,2$, be two sets of data as in theorem 1.4 with the properties a), b), and c). If $\varepsilon = \inf(\varepsilon_1, \varepsilon_2)$ and $V = U_1 \cap U_2$, then $\Phi_1 = \Phi_2$ on $(\tau-\varepsilon, \tau+\varepsilon) \times V$.

1.5. <u>COROLLARY</u>. There exist an open neighbourhood W of $\{0\} \times M$ in $\mathbb{R} \times M$ and a map Φ of class C^r (respectively analytic) of W into M with the following properties satisfied at every point y of M:

a) $\mathbb{R} \times \{y\} \cap W$ is connected;

b) $t \to \Phi(t,y)$ is an integral curve of X;

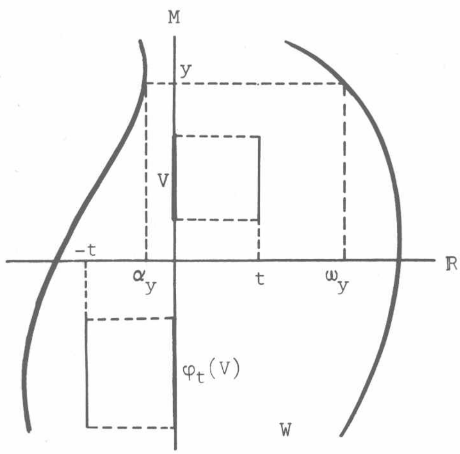
c) $\Phi(0,y) = y$;

d) if (t',y), $(t+t',y)$ and $(t,\Phi(t',y))$ are in W, then $\Phi(t+t',y)=\Phi(t,\Phi(t',y))$.

Furthermore, if W_i, Φ_i, $i=1,2$, are two such data satisfying a), b), and c), then they also satisfy d), and $\Phi_1 = \Phi_2$ on $W_1 \cap W_2$.

Continuing with these notations we let V be an open set of M such that $\{t\}\times V$ and $\{-t\}\times \Phi(\{t\}\times V)$ are contained in W. Then $\Phi(\{t\}\times V)$ is open in M, and the map φ_t: $x\mapsto\Phi(t,x)$ is a diffeomorphism of V onto this open set having the inverse φ_{-t}: $z\mapsto\Phi(-t,z)$.

In addition, for $\{t'\}\times V$, $\{t+t'\}\times V$, and $\{t\}\times \Phi(\{t'\}\times V)$ being contained in W as well, we have $\varphi_{t+t'}=\varphi_t\circ\varphi_{t'}$ on V. These remarks and the considerations in §1.7 justify the following definition:



1.6. DEFINITION. <u>A local one-parameter group of diffeomorphisms (or a flow)</u> of class C^r (respectively analytic) of M is an ordered pair (W,Φ), where W is an open neighbourhood of $\{0\}\times M$ in $\mathbb{R}\times M$ and Φ a map of class C^r (analytic) of W into M, having at every point y of M the following properties:

a) $\mathbb{R}\times\{y\}\cap W$ is connected;

b) $\Phi(0,y) = y$;

c) if (t',y), $(t+t',y)$, and $(t, \Phi(t',y))$ are in W then $\Phi(t+t',y) = \Phi(t, \Phi(t',y))$.

1.7. For $W = \mathbb{R}\times M$, Φ is a (global) <u>one-parameter group of diffeomorphisms</u> of M. In this case the map $\varphi_t : x \longmapsto \Phi(t,x)$ is a diffeomorphism of M for every $t\in\mathbb{R}$, and we have:

i) φ_o = identity map;

ii) $\varphi_{t+t'} = \varphi_t \circ \varphi_{t'}$;

iii) $(\varphi_t)^{-1} = \varphi_{-t}$.

In other words, Φ is a differentiable (analytic) action of $\mathbb{R}$ on M.

Such a flow will often be denoted by $(\varphi_t)_{t\in\mathbb{R}}$.

1.8. <u>Remark</u>. A vector field X on M allows the construction of a flow (W,Φ) of diffeomorphisms of M (of the same differentiability class as X) such that for every point y of M the curve $t\longmapsto\Phi(t,y)$ is an integral curve of X. (W,Φ) is a local one-parameter group <u>generated</u> by X.

The germ of such a flow along $\{0\}\times M$ is uniquely determined by X (corollary 1.5).

Conversely, if (W,Φ) is a flow of class C^r (respectively analytic) on M, there exists one and only one generating vector field X of class C^{r-1} (respectively analytic): the value of X at a point $y\in M$ is the vector tangent in y to the curve $t\longmapsto\Phi(t,y)$. This field whose value at the point y is $\frac{\partial\Phi}{\partial t}(0,y)$ is of class C^{r-1}. It is easy to see, by using properties of flows, that its integral curves are the maps $t\longmapsto\Phi(t,y)$.

1.9. <u>PROPOSITION</u>. The set of all flows generated by the vector field X, ordered by inclusion, has a unique maximal element.

The "union" of all flows generated by X is indeed itself a flow generated by X.

If this local one-parameter group is a global group, X is called a complete vector field.

1.10. Examples.

i) The vector field $X = \sum_i x_i \frac{\partial}{\partial x_i}$ is complete on $\mathbb{R}^m$: it generates the flow of all homotheties of $\mathbb{R}^m$ (the diffeomorphism φ_t is the homothety of ratio e^t).

ii) The maximal flow on $\mathbb{R}$ generated by the field $X = x^2 \frac{\partial}{\partial x}$ is given by

$$W = \left\{(t,x) \in \mathbb{R}\times\mathbb{R} \;\middle|\; 1-tx > 0\right\},$$

$$(t,x) = \frac{x}{1 - tx} .$$

1.11. Remarks.

i) In the case of the covering map of example iv) of 1.2, the two vector fields X and $\tilde{X}$ are simultaneously complete or not complete.

ii) If the vector field X is complete and generates a flow $(\varphi_t)_{t\in\mathbb{R}}$ on M, one has $\varphi_t^T \circ X \circ \varphi_{-t} = X$ for every $t \in \mathbb{R}$: every diffeomorphism φ_t transforms any integral curve of X into another one.

It follows that for a submanifold N of codimension 1 of M which is transverse*) to X the submanifold $\varphi_t(N)$ remains transverse to X for every $t \in \mathbb{R}$. Hence the map $(t,x) \longrightarrow \varphi_t(x)$ is then a submersion of $\mathbb{R} \times N$ into M.

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* A submanifold N of codimension 1 is transverse to X if for every point y of N the vector X(y) does not belong to the tangent hyperplane $T_y(N)$ to N at y.

If dimension $M = 2$, and if N is diffeomorphic to the circle S^1 (or an interval of $\mathbb{R}$) one calls N a closed transversal (respectively a transverse arc) to X.

iii) Let Y be a complete vector field on the product $M \times \mathbb{R}$ of the form $Z(y,s) + \frac{\partial}{\partial s}$, $Z(y,s) \in T_yM$. Then the flow generated on $M \times \mathbb{R}$ may be written as $(t,y,s) \longmapsto (f_{t,s}(y), t+s)$, where $f_{t,s}$ is a family of diffeomorphisms of M satisfying the following properties:

a) $f_{0,s}$ = identity map for every s;

b) $f_{t+t',s} = f_{t,t'+s} \circ f_{t',s}$;

c) $(f_{t,s})^{-1} = f_{-t,t+s}$.

Letting $g_{t,s} = f_{t-s,s}$ we thus obtain a family of diffeomorphisms of M with the following properties:

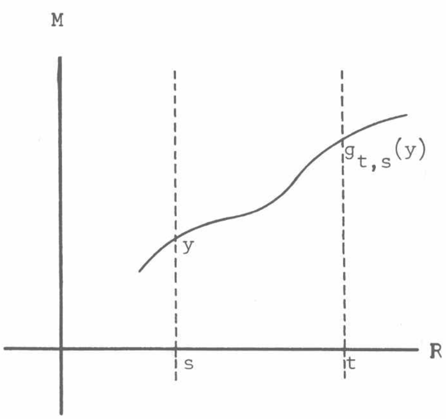
a) $g_{s,s}$ = identity map for every s ;

b) $g_{t,s} \circ g_{s,r} = g_{t,r}$;

c) $(g_{t,s})^{-1} = g_{s,t}$.

This family is therefore determined by the "isotopy of the identity" $h_t = g_{t,0}$, because of $g_{t,s} = h_t \circ h_s^{-1}$.

Furthermore, keeping y and s fixed, the curve $t \longmapsto g_{t,s}(y)$ is the solution of the non-autonomous system $z' = Z(z,t)$ with initial condition $z(s) = y$. The diffeomorphism $g_{t,s}$ may thus be interpreted as the translation of $M \times \{s\}$ to $M \times \{t\}$ along the integral curves of Y.



If Z is periodic with period τ we have $g_{t+\tau,s+\tau} = g_{t,s}$. Then the field Y induces a vector field on the cylinder $M \times S^1 = M \times (\mathbb{R}/\tau\mathbb{Z})$ (cf. exercise ii) of II-1.12).

1.12. THEOREM. A vector field with compact support is complete. (The support of a vector field is the closure of the set of its regular points.)

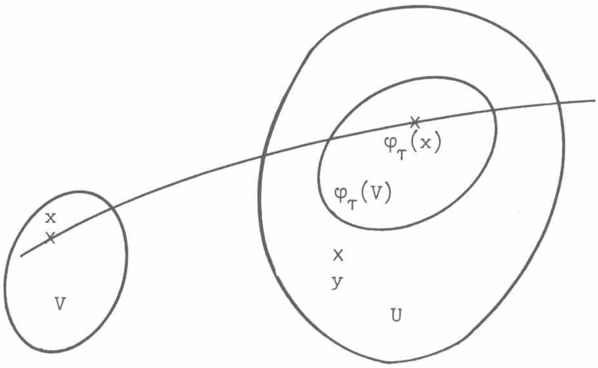
In particular every vector field on a compact manifold is complete.

Theorem 1.12 is an immediate consequence of the following lemma:

1.13. LEMMA. Let (W, Φ), with $W = \bigcup_{y \in M} (\alpha_y, \omega_y) \times y$, be the maximal flow generated by X, and let the curve $\Phi([0, \omega_x) \times x)$ be relatively compact for a certain point x of M. Then $\omega_x = +\infty$.

Proof.

Assume the curve $\Phi([0, \omega_x) \times \{x\})$ to be relatively compact and ω_x finite, and let y be a point of accumulation of $\Phi(t,x)$ for t tending towards ω_x.

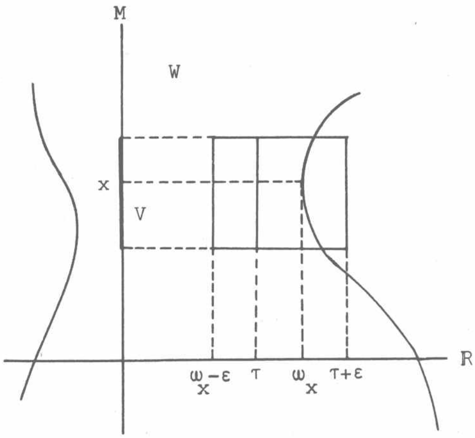


We choose an open neighbourhood U of y, a number ε between 0 and ω_x, and a differentiable map $\Psi: (-\varepsilon, +\varepsilon) \times U \to M$ having the properties a), b), and c) of theorem 1.4.

Let $\tau \in (\omega_x - \varepsilon, \omega_x)$ such that $\Phi(\tau, x) \in U$, and let V be an open neighbourhood of x such that $\{\tau\} \times V$ is contained in W, and $\Phi(\{\tau\} \times V)$ in U.

Under these conditions Φ can be extended to the open set $W \cup ((\omega_x - \varepsilon, \tau + \varepsilon) \times V)$ by defining $\Phi(t,z) = \Psi(t-\tau, \Phi(\tau,z))$ for $z \in V$ and $t \in (\omega_x - \varepsilon, \tau + \varepsilon)$.

We thus obtain a new flow generated by X and strictly greater than (W, Φ) which is contrary to the assumption of maximality. Q.E.D.



Still denoting by (W, Φ) the maximal flow generated by X we can draw the following conclusions from lemma 1.13:

1.14 <u>PROPOSITION</u>. Every integral curve of X assuming the value y for $t = \tau$ is the restriction of the map $t \longmapsto \Phi(t-\tau, y)$ to a subinterval of $(\alpha_y - \tau, \omega_y - \tau)$.

1.15 <u>COROLLARY</u>. A vector field X is complete if and only if every integral curve of X can be extended to an integral curve defined on all of $\mathbb{R}$.

1.16 PROPOSITION. Let Y be a vector field on the product $M \times \mathbb{R}$ of the form $Z(y,s) + \frac{\partial}{\partial s}$, with $Z(y,s) \in T_yM$, and let the projection onto M of the support of Z be relatively compact. Then the field Y is complete.

1.17. In view of proposition 1.14 the curve c_y defined on the interval (α_y, ω_y) by $t \mapsto \Phi(t,y)$ is called the maximal integral curve of X passing through y.

The images of any two maximal integral curves are either disjoint or they coincide. The set of these images defines thus a partition of M; its elements are called the orbits of X. In particular the singular points of X are point orbits. All other orbits are called non-singular. The quotient space of M by the partition of orbits of X is called the orbit space of X.

1.18. Classification of Orbits. A first classification of the orbits of the vector field X may be obtained by observing that, for a given orbit γ, all maximal integral curves c_y, $y \in \gamma$, which parametrize γ, satisfy simultaneously one of the following three properties:

i) c_y is injective;

ii) c_y is neither injective nor constant;

iii) c_y is constant.

In the first case the map c_y is an injective immersion of the interval (α_y, ω_y) into M; but it is not necessarily an embedding: there exist, e.g. on the torus T^2, vector fields all of whose orbits are everywhere dense (cf. example 4.12).

In the second case, if $c_y(b) = c_y(a)$, $b > a$, the map c_y is defined on $\mathbb{R}$ (lemma 1.13), and is periodic with a minimal period τ which is a fraction of b-a (this period is evidently independent of the choice of the point y on γ). γ is then called a <u>periodic orbit</u> of X, of period τ; it is a sub-manifold of M diffeomorphic to the circle S^1.

Finally, in the third case, y is a singular point of X.

1.19. <u>PROPOSITION</u>. Let X be a vector field on the paracompact manifold M. There exists a strictly positive function f on M, of the same differentiability class as X, such that the vector field Y = fX is complete.

<u>Proof</u>. Paracompactness of M implies the existence of a proper function g of class C^s on M. Let $f = \exp(-(Xg)^2)$. If Y = fX, then $|Yg| = |(Xg)\exp(-(Xg)^2)| \leq 1$ on M. If c denotes an integral curve of Y defined on a bounded interval J, then $\frac{d}{dt}(g \circ c) = (Yg) \circ c$; hence $\left|\frac{d}{dt}(g \circ c)\right| \leq 1$ on J.

Thus the image of $g \circ c$ is bounded and hence the image of c is relatively compact. The proof is concluded by an application of lemma 1.13.

Q.E.D.

1.20. <u>Remark</u>. If c is the maximal integral curve of the vector field X passing through z, and if f is a never vanishing function on M, then the maximal integral curve of Y = fX through z is the map $t \mapsto c(h(t))$, where h is the maximal solution of the differential equation $\frac{ds}{dt} = f(c(s))$ satisfying $h(0) = 0$. Thus these maximal integral curves differ only by a change of parameter, which preserves the orientation for positive f. Hence the orbits of X and Y coincide. We may thus assume, in what follows, the field to be complete (as far as properties of the orbits of a vector field on a paracompact manifold which are invariant with respect to parameter transformations are concerned).

As an example we have

1.21. PROPOSITION. The equivalence relation on M whose classes are the orbits of a vector field is open.

Indeed, denoting by $(\varphi_t)_{t \in R}$ the one-parameter group of diffeomorphisms of M generated by the given complete vector field, and by U an open set of M, then $\bigcup_{t \in \mathbb{R}} \varphi_t(U)$ (the "saturated set" of U) is open.

1.22. Remark. Some of the preceding results may be extended to the case where M is a manifold with boundary, provided that the vector field X has a "sufficiently nice" behaviour on the boundary of M; for example if it is either tangent or transverse to the boundary.

In particular, if X is a vector field on M transverse to the boundary and pointing inward along the boundary, one obtains, by restriction to positive times, a local one-parameter semigroup of diffeomorphisms of M.

Using a partition of unity in the case of a paracompact manifold M one can construct such an inward pointing vector field which is "positively complete". Hence:

1.23. THEOREM. Let M be a paracompact manifold with boundary. There exists a diffeomorphism h of $\partial M \times [0,+\infty)$ on an open set V of M satisfying $h(y,0) = y$ for all $y \in \partial M$.

V is then called a collar of the boundary of M.

1.24. Exercise. Let M and N be two compact manifolds with boundary of the same dimension, and let h be a diffeomorphism of a component of the boundary of M onto a component of the boundary of N. Then there exists on the adjunction space $V = M \cup_h N$ (obtained by glueing M to N