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Abstract

In this paper we introduce the notion of a geometric associative r-matrix attached to a genus one fibration with a section and irreducible fibres. It allows us to study degenerations of solutions of the classical Yang–Baxter equation using the approach of Polishchuk. We also calculate certain solutions of the classical, quantum and associative Yang–Baxter equations obtained from moduli spaces of (semi-)stable vector bundles on Weierstraß cubic curves.

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Contents

Introduction

There are many indications (for example from homological mirror symmetry) that the formalism of derived categories provides a compact way to formulate and solve complicated non-linear analytical problems. However, one would like to have more concrete examples, in which one can follow the full path starting from a categorical set-up and ending with an analytical output. In this article we study the interplay between the theory of the associative, classical and quantum Yang–Baxter equations and properties of vector bundles on projective curves of arithmetic genus one, following the approach of Polishchuk [**57**].

Let $\mathfrak{g}$ be the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ and $A = U(\mathfrak{g})$ its universal enveloping algebra. The classical Yang–Baxter equation (CYBE) is

$$[r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0,$$

where $r(z)$ is the germ of a meromorphic function of one variable in a neighbourhood of 0 taking values in $\mathfrak{g} \otimes \mathfrak{g}$. The upper indices in this equation indicate various embeddings of $\mathfrak{g} \otimes \mathfrak{g}$ into $A \otimes A \otimes A$. For example, the function r^{13} is defined as

$$r^{13} : \mathbb{C} \xrightarrow{r} \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\tau_{13}} A \otimes A \otimes A,$$

where $\tau_{13}(x \otimes y) = x \otimes 1 \otimes y$. Two other maps r^{12} and r^{23} have a similar meaning.

In the physical literature, solutions of (CYBE) are frequently called *r-matrices*. They play an important role in mathematical physics, representation theory, integrable systems and statistical mechanics.

By a famous result of Belavin and Drinfeld [**8**], there exist exactly three types of non-degenerate solutions of the classical Yang–Baxter equation: elliptic (two-periodic), trigonometric (one-periodic) and rational. This trichotomy corresponds to three models in statistical mechanics: XYZ (elliptic), XXZ (trigonometric) and XXX (rational), see [**7**].

Belavin and Drinfeld have also obtained a complete classification of elliptic and trigonometric solutions, see [**8**, Proposition 5.1 and Theorem 6.1]. A certain classification of rational solutions was given by Stolin [**63**, Theorem 1.1].

This article is devoted to a study of degenerations of elliptic r-matrices into trigonometric and then into rational ones. We hope that this sort of questions will be interesting from the point of view of applications in mathematical physics. In order to attack this problem we use a construction of Polishchuk [**57**]. After certain modifications of his original presentation, the core of this method can be described as follows.

Let E be a Weierstraß cubic curve, $\breve{E} \subset E$ the open subset of smooth points, $M = M_E^{(n,d)}$ the moduli space of stable bundles of rank n and degree d, assumed to be coprime. Let $\mathcal{P} = \mathcal{P}(n,d) \in \mathsf{VB}(E \times M)$ be a universal family of the moduli functor $\underline{\mathsf{M}}_E^{(n,d)}$. For a point $v \in M$ we denote by $\mathcal{V} = \mathcal{P}\big|_{E\times v}$ the corresponding vector bundle on E. Consider the following data:

- two distinct points $v_1, v_2 \in M$ in the moduli space;
- two distinct points $y_1, y_2 \in \breve{E}$ such that $\mathcal{V}_1(y_2) \not\cong \mathcal{V}_2(y_1)$.

Using Serre Duality, the triple Massey product

$$\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Ext}^1_E(\mathbb{C}_{y_1}, \mathcal{V}_2) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2}),$$

induces a linear morphism

$$r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2} : \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \longrightarrow \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_2})$$

which satisfies the so-called *associative Yang–Baxter equation* (AYBE)

$$\left(r^{\mathcal{V}_3,\mathcal{V}_2}_{y_1,y_2}\right)^{12}\left(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_1,y_3}\right)^{13} - \left(r^{\mathcal{V}_1,\mathcal{V}_3}_{y_2,y_3}\right)^{23}\left(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_2}\right)^{12} + \left(r^{\mathcal{V}_1,\mathcal{V}_2}_{y_1,y_3}\right)^{13}\left(r^{\mathcal{V}_2,\mathcal{V}_3}_{y_2,y_3}\right)^{23} = 0$$

viewed as a map

$$\begin{array}{c}\mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_3}) \\ \big\downarrow \\ \mathsf{Hom}_E(\mathcal{V}_2, \mathbb{C}_{y_1}) \otimes \mathsf{Hom}_E(\mathcal{V}_3, \mathbb{C}_{y_2}) \otimes \mathsf{Hom}_E(\mathcal{V}_1, \mathbb{C}_{y_3}).\end{array}$$

This morphism can be rewritten as the germ of a tensor-valued meromorphic function in four variables, defined in a neighbourhood of a smooth point o of the moduli space $M \times M \times E \times E$ (the choice of o will be explained in Corollary 3.2.13)

$$r(\mathcal{V}_1, \mathcal{V}_2; y_1, y_2) : (\mathbb{C}^2 \times \mathbb{C}^2, 0) \cong \big((M\times M)\times(E\times E), o\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})\otimes\mathsf{Mat}_{n\times n}(\mathbb{C}).$$

Since the complex manifold $M_E^{(n,d)}$ is a homogeneous space over the algebraic group $J = \mathsf{Pic}^0(E)$, it turns out that

$$r(v_1, v_2; y_1, y_2) \sim r(v_1 - v_2; y_1, y_2) = r(v; y_1, y_2),$$

with respect to a certain equivalence relation on the set of solutions. We show that this equivalence relation corresponds to a change of a trivialisation of the universal family $\mathcal{P}$.

Let e be the neutral element of J. Polishchuk has shown [**57**, Lemma 1.2] that the function of two variables

$$\bar{r}(y_1, y_2) = \lim_{v\to e} (\mathrm{pr} \otimes \mathrm{pr}) r(v; y_1, y_2) \in \mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$$

is a non-degenerate unitary solution of the classical Yang–Baxter equation. Moreover, under certain restrictions (which are always fulfilled at least for elliptic curves and Kodaira cycles of projective lines), for any fixed value $g \neq e$ from a small neighbourhood $U_e \subseteq J$ of e, the tensor-valued function

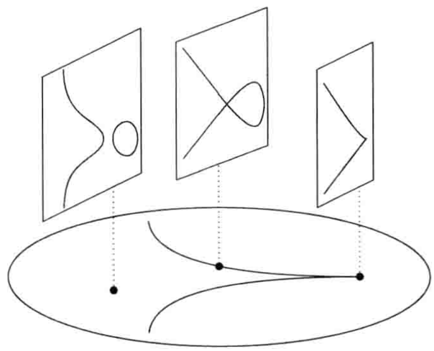
$$r : \big(\{g\} \times E \times E, e\big) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

satisfies the *quantum* Yang–Baxter equation, see [**58**, Theorem 1.4]. Hence, this approach gives an explicit method to quantise some known solutions of the classical Yang–Baxter equation.

Moreover, as was pointed out by Kirillov [**39**], a solution $r(v; y_1, y_2)$ of the associative Yang–Baxter equation defines an interesting family of pairwise commuting first order differential operators, generalising Dunkl operators studied by Buchstaber, Felder and Veselov [**14**], see Proposition 1.2.6.

The aim of our article is to study a *relative* version of Polishchuk's construction. Although most of the results can be generalised to the case of arbitrary reduced projective curves of arithmetic genus one having trivial dualising sheaf, in this article we shall concentrate mainly on the case of irreducible curves.

Let E be a Weierstraß cubic curve, i.e. a plane projective curve given by the equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$. It is singular if any only if $\Delta := g_2^3 - 27g_3^2 = 0$.



Unless $g_2 = g_3 = 0$, the singularity is a node, whereas for $g_2 = g_3 = 0$ it is a cusp.

A connection between the theory of vector bundles on cubic curves and exactly solvable models of mathematical physics was observed a long time ago, see for example [**46**, Chapter 13] and [**49**] for a link with KdV equation, [**23**] for applications to integrable systems and [**10**] for an interplay with Calogero-Moser systems. In particular, the correspondence

elliptic	elliptic
trigonometric	nodal
rational	cuspidal

was discovered at the very beginning of the algebraic theory of completely integrable systems.

In this article we follow another strategy. Instead of looking at each curve of arithmetic genus one individually, we consider the relative case, so that all solutions will be considered as specialisations of one *universal* solution. Our main result can be stated as follows.

THEOREM. *Let $E \to T$ be a genus one fibration with a section having reduced and irreducible fibres, $M = M_{E/T}^{(n,d)}$ the moduli space of relatively stable vector bundles of rank n and degree d. We construct a meromorphic function*

$$r : (M \times_T \times M \times_T E \times_T E, o) \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$$

in a neighbourhood of a smooth point o of $M \times_T \times M \times_T E \times_T E$, which satisfies the associative Yang–Baxter equation for each fixed value $t \in T$ and $(v_1, v_2, y_1, y_2) \in \big((M_{E_t} \times M_{E_t}) \times (E_t \times E_t), o\big)$. Moreover, $r_t(v_1, v_2, y_1, y_2)$ is analytic outside the hypersurfaces $v_1 = v_2$ and $y_1 = y_2$, and it is compatible with base change of the given family $E \to T$. The corresponding solution $\bar{r}_t(y)$ of the classical Yang–Baxter equation is

- *elliptic if E_t is smooth;*
- *trigonometric if E_t is nodal;*
- *rational if E_t is cuspidal.*

We also carry out explicit calculations for vector bundles of rank two and degree one on irreducible Weierstraß cubic curves. In the case of an elliptic curve $E = E_\tau$ the corresponding solution is

$$r_{\mathrm{ell}}(v;y) = \frac{\theta_1'(0|\tau)}{\theta_1(y|\tau)}\left[\frac{\theta_1(y+v|\tau)}{\theta_1(v|\tau)}\mathbb{1}\otimes\mathbb{1} + \frac{\theta_2(y+v|\tau)}{\theta_2(v|\tau)}h\otimes h + \right.$$
$$\left. + \frac{\theta_3(y+v|\tau)}{\theta_3(v|\tau)}\sigma\otimes\sigma + \frac{\theta_4(y+v|\tau)}{\theta_4(v|\tau)}\gamma\otimes\gamma\right],$$

where $\mathbb{1} = e_{11} + e_{22}$, $h = e_{11} - e_{22}, \sigma = i(e_{21} - e_{12})$ and $\gamma = e_{21} + e_{12}$.

In the case of a nodal cubic curve we obtain

$$r_{\mathrm{trg}}(v;y) = \frac{\sin(y+v)}{\sin(y)\sin(v)}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22}) + \frac{1}{\sin(v)}(e_{11}\otimes e_{22} + e_{22}\otimes e_{11}) +$$
$$+ \frac{1}{\sin(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \sin(y+v)e_{21}\otimes e_{21}$$

and in the case of a cuspidal cubic curve, the associative r-matrix is

$$r_{\mathrm{rat}}(v;y_1,y_2) = \frac{1}{v}\mathbb{1}\otimes\mathbb{1} + \frac{2}{y_2 - y_1}(e_{11}\otimes e_{11} + e_{22}\otimes e_{22} + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) +$$
$$+ (v - y_1)e_{21}\otimes h + (v + y_2)h\otimes e_{21} - v(v - y_1)(v + y_2)e_{21}\otimes e_{21}.$$

Our results imply that up to a gauge transformation the trigonometric and rational solutions $r_{\mathrm{trg}}(v;y)$ and $r_{\mathrm{rat}}(v;y_1,y_2)$ are degenerations of $r_{\mathrm{ell}}(v;y)$, which seems to be difficult to show by a direct computation.

Moreover, for generic v the tensors $r_{\mathrm{ell}}(v;y)$, $r_{\mathrm{trg}}(v;y)$ and $r_{\mathrm{rat}}(v;y_1,y_2)$ satisfy the quantum Yang–Baxter equation and are quantisations of the following classical r-matrices:

- Elliptic solution found and studied by Baxter, Belavin and Sklyanin:

$$\bar{r}_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)}h\otimes h + \frac{1+\mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \frac{1-\mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12}\otimes e_{12} + e_{21}\otimes e_{21}).$$

- Trigonometric solution of Cherednik:

$$\bar{r}_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y)h\otimes h + \frac{1}{\sin(y)}(e_{12}\otimes e_{21} + e_{21}\otimes e_{12}) + \sin(y)e_{21}\otimes e_{21}.$$

- Rational solution

$$\bar{r}_{\mathrm{rat}}(y) = \frac{1}{y}\Big(\frac{1}{2}h\otimes h + e_{12}\otimes e_{21} + e_{21}\otimes e_{12}\Big) + y(e_{21}\otimes h + h\otimes e_{21}) - y^3 e_{21}\otimes e_{21},$$

which is gauge equivalent to a solution found by Stolin [**63**].

This paper is organised as follows. In Chapter 1 we collect some results about the associative Yang–Baxter equation and its relations with Dunkl operators as well as with the classical and quantum Yang–Baxter equations.

Chapter 2 consists of two sections. Section 2.1 gives a short introduction into a construction of Polishchuk which provides a method to obtain solutions of Yang–Baxter equations from triple Massey products in a derived category. In order to be able to calculate solutions explicitly, this construction has to be translated into another language, involving residue maps. In Section 2.2 we explain the corresponding result of Polishchuk whereby we provide some details which are only implicit in [**57**]. The understanding of these details is crucial for the study of the relative case, which is carried out in Chapter 3.

Theorem 3.2.13 is the main result of this article. It states that for any genus one fibration $E \to T$ satisfying certain restrictions and any pair of coprime integers $0 < d < r$ one can attach a family of solutions of the associative Yang–Baxter equation $r^{\xi}(v_1, v_2; y_1, y_2)$ depending *analytically* on the parameter of the basis and functorial with respect to the base change. This solutions actually depend on the choice of a trivialisation ξ of the universal family $\mathcal{P}(n, d)$ of stable vector bundles of rank n and degree d. However, in Proposition 3.2.12 we show that the choice of another trivialisation ζ leads to a *gauge equivalent* solution $r^{\zeta}(v_1, v_2; y_1, y_2)$

In Section 3.3 we prove that in the case of a Weierstraß cubic curve E there exists a trivialisation ξ of the universal family $\mathcal{P}(n, d)$ such that the corresponding solution $r^{\xi}(v_1, v_2; y_1, y_2)$ is invariant under simultaneous shifts

$$v_1 \mapsto v_1 + v, \quad v_2 \mapsto v_2 + v.$$

In other words, the solution $r^{\xi}(v_1, v_2; y_1, y_2)$, also called the *geometric associative r-matrix*, depends on the difference $v_2 - v_1$ of the first pair of spectral parameters only. Hence, the obtained solution $r^{\xi}(v; y_1, y_2)$ also satisfies the quantum Yang–Baxter equation and defines an interesting quantum integrable system. The key point of the proof is to show equivariance of triple Massey products with respect to the action of the Jacobian J on the moduli space $M_E^{(n,d)}$.

Since it is indispensable for carrying out explicit calculations of r-matrices, in the following chapters we elaborate foundations of the theory of vector bundles on genus one curves. In Chapter 4 we recall some classical results about holomorphic vector bundles on a smooth elliptic curve. Using the methods described before, we explicitly compute the solution of the associative Yang–Baxter equation and the classical r-matrix corresponding to a universal family of stable vector bundles of rank two and degree one. These solutions were obtained by Polishchuk in [**57**, Section 2] using homological mirror symmetry and formulae for higher products in the Fukaya category of an elliptic curve. Our direct computation, however, is independent of homological mirror symmetry. We are lead directly to express the resulting associative r-matrix in terms of Jacobi's theta-functions and the corresponding classical r-matrix in terms of the elliptic functions $\mathrm{sn}(z)$, $\mathrm{cn}(z)$ and $\mathrm{dn}(z)$.

Chapter 5 is devoted to similar calculations for nodal and cuspidal Weierstraß curves. Our computations are based on the description of vector bundles on singular genus one curves in terms of so-called matrix problems, which was given by Drozd and Greuel [**25**] and the first-named author [**15**]. We show that their description of canonical forms of matrix problems corresponds precisely to a very explicit presentation of universal families of stable vector bundles. We explicitly compute geometric r-matrices coming from universal families of stable vector bundles of rank two and degree one on a nodal and a cuspidal cubic curves and the r-matrix coming from the universal family of semi-stable vector bundles of rank two and degree zero on a nodal cubic curve.

We conclude with a brief summary of the analytical results in Chapter 6.

NOTATION. Throughout this paper we work in the category of analytic spaces over the field of complex numbers $\mathbb{C}$, see [**55**]. However, most of the results remain valid in the category of algebraic varieties over an algebraically closed field $\boldsymbol{k}$ of characteristic zero.

If V, W are two complex vector spaces, $\mathsf{Lin}(V, W)$ denotes the vector space of complex linear maps from V to W.

For an additive category C, a pair of objects $X, Y \in \mathsf{C}$ and two isomorphisms $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ we denote by $\mathrm{cnj}(f, g)$ the morphism of abelian groups $\mathsf{Hom}_{\mathsf{C}}(X, Y) \longrightarrow \mathsf{Hom}_{\mathsf{C}}(X', Y')$ mapping a morphism h to the composition $g \circ h \circ f^{-1}$.

If X is a complex projective variety, we denote by $\mathsf{Coh}(X)$ the category of coherent $\mathcal{O}_X$-modules and by $\mathsf{VB}(X)$ its full subcategory of locally free sheaves (holomorphic vector bundles). The torsion sheaf of length one, supported at a closed point $y \in X$, is always denoted by $\mathbb{C}_y$. By $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ we denote the full subcategory of the derived category of the abelian category of all $\mathcal{O}_X$-modules whose objects are those complexes which have bounded and coherent cohomology. The notation $\mathsf{Perf}(X)$ is used for the full subcategory of $\mathsf{D}^{\mathsf{b}}_{\mathsf{coh}}(X)$ whose objects are isomorphic to bounded complexes of locally free sheaves. For a morphism of reduced complex spaces $E \xrightarrow{p} T$ we denote by $\breve{E}$ the regular locus of p.

A Weierstraß curve is a plane cubic curve given in homogeneous coordinates by an equation $zy^2 = 4x^3 - g_2xz^2 - g_3z^3$, where $g_1, g_2 \in \mathbb{C}$. Such a curve is always irreducible. It is a smooth elliptic curve if and only if $\Delta(g_2, g_3) = g_2^3 - 27g_3^2 \neq 0$.

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CHAPTER 1

Yang–Baxter Equations

1.1. The classical Yang–Baxter equation

In this section we recall some standard results about Yang–Baxter equations. Let $\mathfrak{g}$ be a simple complex Lie algebra and $\langle\ ,\ \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form. Throughout this paper $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. The classical Yang–Baxter equation (CYBE) is

$$[r^{12}(y_1,y_2), r^{23}(y_2,y_3)] + [r^{12}(y_1,y_2), r^{13}(y_1,y_3)] + [r^{13}(y_1,y_3), r^{23}(y_2,y_3)] = 0, \tag{1.1}$$

where $r(x,y)$ is the germ of a meromorphic function of two complex variables in a neighbourhood of 0, taking values in $\mathfrak{g} \otimes \mathfrak{g}$. A solution of (1.1) is called *unitary* if

$$r^{12}(y_1,y_2) = -r^{21}(y_2,y_1)$$

and *non-degenerate* if $r(y_1,y_2) \in \mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g}^* \otimes \mathfrak{g} \cong \mathsf{End}(\mathfrak{g})$ is invertible for generic (y_1,y_2). On the set of solutions of (1.1) there exists a natural action of the group of holomorphic function germs $\phi : (\mathbb{C},0) \longrightarrow \mathrm{Aut}(\mathfrak{g})$ given by the rule

$$r(y_1,y_2) \mapsto \big(\phi(y_1) \otimes \phi(y_2)\big) r(y_1,y_2). \tag{1.2}$$

PROPOSITION 1.1.1 (see [**9**]). *Modulo the equivalence relation (1.2) any non-degenerate unitary solution of the equation (1.1) is equivalent to a solution $r(u,v) = r(u-v)$ depending on the difference (or the quotient) of spectral parameters only.*

This means that equation (1.1) is essentially equivalent to the equation

$$[r^{12}(x), r^{13}(x+y)] + [r^{13}(x+y), r^{23}(y)] + [r^{12}(x), r^{23}(y)] = 0. \tag{1.3}$$

Although the classical Yang–Baxter equation with one spectral parameter is better adapted for applications in mathematical physics, it seems that from a geometric point of view equation (1.1) is more natural.

Let $m = \dim(\mathfrak{g})$, $e_1, e_2, \dots, e_m$ be a basis of $\mathfrak{g}$ and $e^1, e^2, \dots, e^m$ be the dual basis of $\mathfrak{g}$ with respect to the Killing form $\langle\ ,\ \rangle$. Then $\Omega = \sum_{i=1}^m e^i \otimes e_i \in \mathfrak{g} \otimes \mathfrak{g}$ is independent of the choice of a basis and is called the *Casimir element.*

THEOREM 1.1.2 (see Proposition 2.1 and Proposition 4.1 in [**8**]). *Let $r(y)$ be a non-constant non-degenerate solution of (1.3). Then the tensor $r(y)$*

- *has a simple pole at 0 and $\mathrm{res}_{y=0}\big(r(y)\big) = \alpha\Omega$ for some $\alpha \in \mathbb{C}^*$;*
- *is automatically unitary, i.e. $r^{12}(y) = -r^{21}(-y)$.*

As it was already mentioned in the introduction, there is the following classification of non-degenerate solutions of (CYBE) due to Belavin and Drinfeld.

THEOREM 1.1.3 (see Proposition 4.5 and Proposition 4.7 in [**8**]). *There are three types of non-degenerate solutions of the classical Yang–Baxter equation (1.3): elliptic, trigonometric and rational.*

Let us now consider some examples. Fix the following basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. Note that $\Omega = \frac{1}{2} h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}$ is the Casimir element of $\mathfrak{sl}_2(\mathbb{C})$.

• Historically, the first solution ever found was the rational solution of Yang

$$r_{\mathrm{rat}}(y) = \frac{1}{y}\left(\frac{1}{2} h \otimes h + e_{12} \otimes e_{21} + e_{21} \otimes e_{12}\right).$$

• A few years later, Baxter discovered the trigonometric solution

$$r_{\mathrm{trg}}(y) = \frac{1}{2}\cot(y) h \otimes h + \frac{1}{\sin(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}).$$

• The following solution of elliptic type was found and studied by Baxter, Belavin and Sklyanin:

$$r_{\mathrm{ell}}(y) = \frac{\mathrm{cn}(y)}{\mathrm{sn}(y)} h \otimes h + \frac{1 + \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{21} + e_{21} \otimes e_{12}) + \frac{1 - \mathrm{dn}(y)}{\mathrm{sn}(y)}(e_{12} \otimes e_{12} + e_{21} \otimes e_{21}),$$

where $\mathrm{cn}(y)$, $\mathrm{sn}(y)$ and $\mathrm{dn}(y)$ are doubly periodic meromorphic functions on $\mathbb{C}$ with periods 2 and 2τ. These functions also satisfy identities of the form $f(y+1) = \varepsilon f(y)$ and $f(y+\tau) = \varepsilon f(y)$ with $\varepsilon = \pm 1$.

At first glance, all these solutions seem to be completely different. However, it is easy to see that

$$\lim_{t \to \infty} \frac{1}{t} r_{\mathrm{trg}}\left(\frac{y}{t}\right) = r_{\mathrm{rat}}(y),$$

hence the solution of Yang is a degeneration of Baxter's solution. Moreover, there exist degenerations $\mathrm{dn}(y) \to 1$, $\mathrm{cn}(y) \to \cos(y)$ and $\mathrm{sn}(y) \to \sin(y)$, when the imaginary period τ tends to infinity, see for example [**43**, Section 2.6]. Hence, both solutions of Baxter and Yang are degenerations of the elliptic solution. However, as we shall see later, the theory of degenerations of r-matrices is more complicated as it might look like at first sight.

1.2. The associative Yang–Baxter equation

In this article we deal with a new type of Yang–Baxter equation, called *associative Yang–Baxter equation* (AYBE). It appeared for the first time in a paper of Fomin and Kirillov [**28**]. Later, it was studied by Aguiar [**1**] in the framework of the theory of infinitesimal Hopf algebras. The following version of the associative Yang–Baxter equation with *spectral parameters* is due to Polishchuk [**57**]. A special case of this equation was also considered by Odesski and Sokolov [**54**].

DEFINITION 1.2.1. An associative r-matrix is the germ of a meromorphic function in four variables

$$r : \left(\mathbb{C}^4_{(v_1, v_2; y_1, y_2)}, 0\right) \longrightarrow \mathsf{Mat}_{n \times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n \times n}(\mathbb{C})$$

holomorphic on $\left(\mathbb{C}^4 \setminus V\big((y_1 - y_2)(v_1 - v_2)\big), 0\right)$ and satisfying the equation

$$r(v_1, v_2; y_1, y_2)^{12} r(v_1, v_3; y_2, y_3)^{23} = r(v_1, v_3; y_1, y_3)^{13} r(v_3, v_2; y_1, y_2)^{12} + \\ + r(v_2, v_3; y_2, y_3)^{23} r(v_1, v_2; y_1, y_3)^{13}. \tag{1.4}$$

Such a matrix is called unitary if

$$r(v_1, v_2; y_1, y_2)^{12} = -r(v_2, v_1; y_2, y_1)^{21}. \tag{1.5}$$

On the set of solutions of (1.4) there exists a natural equivalence relation.

DEFINITION 1.2.2 (see Section 1.2 in [**57**]). Let $\phi : (\mathbb{C}^2, 0) \to \mathsf{GL}_n(\mathbb{C})$ be the germ of a holomorphic function and $r(v_1, v_2; y_1, y_2)$ be a solution of (AYBE) then

$$r'(v_1, v_2; y_1, y_2) = \big(\phi(v_1; y_1) \otimes \phi(v_2; y_2)\big) r(v_1, v_2; y_1, y_2) \big(\phi(v_2; y_1)^{-1} \otimes \phi(v_1; y_2)^{-1}\big)$$

is again a solution of (1.4). Two such tensors r and r' are called gauge equivalent. Note that if the matrix r is unitary then r' is unitary, too.

EXAMPLE 1.2.3. Let $r(v_1, v_2; y_1, y_2) \in \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ be a solution of (1.4). If $c \in \mathbb{C}$, the gauge transformation $\phi = \exp(cvy) \cdot \mathbb{1} : (\mathbb{C}^2, 0) \longrightarrow \mathsf{GL}_n(\mathbb{C})$ shows that $\exp\big(c(v_2 - v_1)(y_2 - y_1)\big) r(v_1, v_2; y_1, y_2)$ is a solution of (AYBE), gauge equivalent to $r(v_1, v_2; y_1, y_2)$.

LEMMA 1.2.4. *Let $r(v_1, v_2; y_1, y_2)$ be a unitary solution of the associative Yang–Baxter equation (1.4). Then r also satisfies the "dual" equation*

$$r(v_2, v_3; y_2, y_3)^{23} r(v_1, v_3; y_1, y_2)^{12} = r(v_1, v_2; y_1, y_2)^{12} r(v_2, v_3; y_1, y_3)^{13} + \\ + r(v_1, v_3; y_1, y_3)^{13} r(v_2, v_1; y_2, y_3)^{23}. \tag{1.6}$$

PROOF. Let τ be the linear automorphism of $\mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ defined by $\tau(a \otimes b) = b \otimes a$. Applying the operator $\tau \otimes \mathbb{1}$ to the equation (1.4), we obtain:

$$r(v_1, v_2; y_1, y_2)^{21} r(v_1, v_3; y_2, y_3)^{13} = r(v_1, v_3; y_1, y_3)^{23} r(v_3, v_2; y_1, y_2)^{21} \\ + r(v_2, v_3; y_2, y_3)^{13} r(v_1, v_2; y_1, y_3)^{23}.$$

Using the unitarity condition (1.5) we get:

$$-r(v_2, v_1; y_2, y_1)^{12} r(v_1, v_3; y_2, y_3)^{13} = -r(v_1, v_3; y_1, y_3)^{23} r(v_2, v_3; y_2, y_1)^{12} \\ + r(v_2, v_3; y_2, y_3)^{13} r(v_1, v_2; y_1, y_3)^{23}.$$

After the change of variables $v_1 \leftrightarrow v_2, v_3 \leftrightarrow v_3$ and $y_1 \leftrightarrow y_2, y_3 \leftrightarrow y_3$, we obtain the equation (1.6). □

Assume a unitary solution $r(v_1, v_2; y_1, y_2)$ of the associative Yang–Baxter equation (1.4) depends on the difference $v = v_1 - v_2$ of the first pair of parameters only. For the sake of simplicity, we shall use the notation $r(v_1, v_2; y_1, y_2) = r(v_1 - v_2; y_1, y_2) = r(v; y_1, y_2)$. Then the equation (1.4) can be rewritten as

$$r(u; y_1, y_2)^{12} r(u + v; y_2, y_3)^{23} = r(u + v; y_1, y_3)^{13} r(-v; y_1, y_2)^{12} \\ + r(v; y_2, y_3)^{23} r(u; y_1, y_3)^{13}. \tag{1.7}$$

REMARK 1.2.5. It will be shown in Theorem 3.3.5 that any solution r of the associative Yang–Baxter equation (1.4) obtained from a universal family of stable vector bundles on an irreducible genus one curve, is gauge equivalent to a solution r' depending on the difference $v_1 - v_2$ only.

Let A be the algebra of germs of meromorphic functions $f : \big(\mathbb{C}^4_{(v_1,v_2;w_1,w_2)}, 0\big) \longrightarrow \mathbb{C}$ holomorphic on $\Big(\mathbb{C}^4 \setminus V\big((v_1-v_2)(w_1-w_2)\big), 0\Big)$. A solution of the equation (1.7) defines an element

$$r \in \mathsf{Mat}_{n\times n}(A) \otimes_A \mathsf{Mat}_{n\times n}(A) \cong A \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C}).$$

In a similar way, for any integer $m \ge 3$ denote by B the algebra of germs of meromorphic functions $f : \big(\mathbb{C}^{2m}_{(x_1,\dots,x_m;y_1,\dots,y_m)}, 0\big) \longrightarrow \mathbb{C}$ which are holomorphic on $\big(\mathbb{C}^{2m} \setminus D, 0\big)$, where D is the divisor

$$D = V\left(\prod_{i\neq j}(x_i - x_j)(y_i - y_j)\right).$$

Next, for any pair of indices $1 \le i \neq j \le m$ we have

- a ring homomorphism $\psi^{ij} : A \to B$ which sends a function $f(v_1, v_2; w_1, w_2)$ to $f(x_i, x_j; y_i, y_j)$;
- a ring homomorphism $k^{ij} : B \to B$ defined as
$$f(\dots, x_i, \dots, x_j, \dots; \underline{y}) \mapsto f(\dots, x_j, \dots, x_i, \dots; \underline{y});$$
- a morphism $\varrho_{ij} : \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes 2} \longrightarrow \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$ mapping a simple tensor $a \otimes b$ to $1 \otimes \dots 1 \otimes a \otimes 1 \otimes \dots 1 \otimes b \otimes 1 \otimes \dots \otimes 1$, where a and b belong to the i-th and j-th components respectively.

In this notation, consider

$$\Psi^{ij} := \psi_{ij} \otimes \varrho_{ij} \;:\; A \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes 2} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

For example $\Psi^{13}\big(f(v_1, v_2; w_1, w_2) \otimes a \otimes b\big) = f(x_1, x_3; y_1, y_3) \otimes a \otimes 1 \otimes b$. Next, we set $r^{ij} := \Psi^{ij}(r) \in B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$ and

$$K^{ij} = k^{ij} \otimes \mathbb{1} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

Consider the linear operator

$$\tilde{r}^{ij} = r^{ij} \circ K^{ij} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m},$$

which is the composition of K^{ij} and the multiplication with the element r^{ij}. For any $1 \le i \le m$ consider the differential operator

$$\partial_i = \frac{\partial}{\partial x_i} \otimes \mathbb{1} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

Next, for any $\kappa \in \mathbb{C}$ let

$$\theta_i := \kappa\partial_i + \sum_{j\neq i} \tilde{r}^{ij} \;:\; B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}$$

be the *Dunkl operator* of level κ. The following result was explained to the first-named author by Anatoly Kirillov, see also [**39**].

PROPOSITION 1.2.6. *Let $r(v; y_1, y_2) \in \mathsf{Mat}_{n\times n}(A) \otimes_A \mathsf{Mat}_{n\times n}(A)$ be a unitary solution of the equation (1.7), $\kappa \in \mathbb{C}$ be a scalar and θ_i be the Dunkl operator of level κ defined above. Then for all $1 \le i, j \le m$ we have $[\theta_i, \theta_j] = 0$.*

PROOF. First note that

$$\Big(\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_j}\Big) r(x_i - x_j; y_i, y_j) = 0,$$

which implies the equality $[\partial_i + \partial_j, \tilde{r}^{ij}] = 0$. Next, the Yang–Baxter relations (1.4) and (1.6) yield that for any triple of mutually different indices $1 \le i < j < k \le m$ we have

$$\tilde{r}^{ij}\tilde{r}^{jk} = \tilde{r}^{jk}\tilde{r}^{ik} + \tilde{r}^{ik}\tilde{r}^{ij} \quad \text{and} \quad \tilde{r}^{jk}\tilde{r}^{ij} = \tilde{r}^{ik}\tilde{r}^{jk} + \tilde{r}^{ij}\tilde{r}^{ik}.$$

From the unitarity of r it follows that $\tilde{r}^{ij} = -\tilde{r}^{ji}$ for all $1 \le i \ne j \le m$. Finally, the following two equalities are obvious:

$$[\tilde{r}^{ij}, \tilde{r}^{kl}] = 0, \quad [\partial_i, \tilde{r}^{kl}] = 0$$

where $1 \le i, j, k, l \le m$ are mutually distinct. Combining these equalities together, we obtain the claim. □

REMARK 1.2.7. The above proposition means that to any unitary solution of the associative Yang–Baxter equation (1.7) one can attach a very interesting second order differential operator

$$H = \theta_1^2 + \theta_2^2 + \cdots + \theta_m^2 : \quad B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m} \longrightarrow B \otimes_{\mathbb{C}} \mathsf{Mat}_{n\times n}(\mathbb{C})^{\otimes m}.$$

These operators are "matrix versions" of the Hamiltonians considered in the work of Buchstaber, Felder and Veselov [**14**].

Another motivation to study solutions of the equation (1.7) is provided by their close connection with the theory of the classical Yang–Baxter equation.

LEMMA 1.2.8 (see Lemma 1.2 in [**57**]). *Let $r(v; y_1, y_2)$ be a unitary solution of the associative Yang–Baxter equation (1.7) and* $\mathrm{pr} : \mathsf{Mat}_{n\times n}(\mathbb{C}) \to \mathfrak{sl}_n(\mathbb{C})$ *be the projection along the scalar matrices, i.e.* $\mathrm{pr}(A) = A - \dfrac{\mathrm{tr}(A)}{n} \cdot \mathbb{1}$. *Assume that* $(\mathrm{pr} \otimes \mathrm{pr})r(v; y_1, y_2)$ *has a limit as $v \to 0$. Then*

$$\bar{r}(y_1, y_2) := \lim_{v\to 0} (\mathrm{pr} \otimes \mathrm{pr}) r(v; y_1, y_2)$$

is a unitary solution of the classical Yang–Baxter equation (1.1).

PROOF. First note that (1.6) implies the equality

$$\begin{aligned} r(v; y_2, y_3)^{23} r(u + v; y_1, y_2)^{12} &= r(u; y_1, y_2)^{12} r(v; y_1, y_3)^{13} \\ &+ r(u + v; y_1, y_3)^{13} r(-u; y_2, y_3)^{23}. \end{aligned}$$

Using the change of variables $u \mapsto -v$ and $v \mapsto u + v$, we obtain the relation

$$\begin{aligned} r(u + v; y_2, y_3)^{23} r(u; y_1, y_2)^{12} &= r(-v; y_1, y_2)^{12} r(u + v; y_1, y_3)^{13} \\ &+ r(u; y_1, y_3)^{13} r(v; y_2, y_3)^{23}. \end{aligned}$$

Subtracting this equation from (1.7) we get

$$\begin{aligned} [r(-v; y_1, y_2)^{12}, r(u + v; y_1, y_3)^{13}] + [r(u; y_1, y_2)^{12}, r(u + v; y_2, y_3)^{23}] + \\ + [r(u; y_1, y_3)^{13}, r(v; y_2, y_3)^{23}] = 0. \end{aligned} \tag{1.8}$$

By definition, the function $r(v; y_1, y_2)$ is meromorphic, hence we can write its Laurent expansion: $r(v; y_1, y_2) = \sum_{\alpha\in\mathbb{Z}} r_\alpha(y_1, y_2)v^\alpha$, where $r_\alpha(y_1, y_2)$ are meromorphic and $r_\alpha = 0$ for $\alpha \ll 0$. Since we have assumed that $(\mathrm{pr}\otimes\mathrm{pr})r(v; y_1, y_2)$ is regular with respect to v in a neighbourhood of $v = 0$, we have $(\mathrm{pr}\otimes\mathrm{pr})r_\alpha(y_1, y_2) = 0$ for all $\alpha \le -1$. This implies, if $\alpha \le -1$, that

$$r_\alpha(y_1, y_2) = s_\alpha(y_1, y_2) \otimes \mathbb{1} + \mathbb{1} \otimes t_\alpha(y_1, y_2)$$

for some matrix-valued functions $s_\alpha(y_1, y_2)$ and $t_\alpha(y_1, y_2)$. Hence,

$$(\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr})\left[r_\alpha^{ij}, r_\beta^{lk}\right] = 0$$

for arbitrary permutations $(ij) \ne (lk)$, all indices $\alpha \le -1$ and $\beta \in \mathbb{Z}$. The claim of Lemma 1.2.8 follows by applying $\mathrm{pr}\otimes\mathrm{pr}\otimes\mathrm{pr}$ to the equation (1.8) and taking the limit $u, v \to 0$. □

A natural question is the following. Let $r = r(v; y_1, y_2)$ be a unitary solution of the associative Yang–Baxter equation (1.7) satisfying the conditions of Lemma 1.2.8 and $s = s(v; y_1, y_2)$ be an equivalent solution in the sense of Definition 1.2.2. Are the corresponding solutions $\bar{r}(y_1, y_2)$ and $\bar{s}(y_1, y_2)$ of the classical Yang–Baxter equation also gauge equivalent?

The answer on this question is affirmative, if one imposes an additional restriction on the function r. Namely, if the Laurent expansion of r has the form:

$$r(v; y_1, y_2) = \frac{\mathbb{1}\otimes\mathbb{1}}{v} + r_0(y_1, y_2) + v r_1(y_1, y_2) + v^2 r_2(y_1, y_2) + \dots\,, \tag{1.9}$$

then the following result is true.

Proposition 1.2.9. *Let $r : (\mathbb{C}^3_{(v;y_1,y_2)}, 0) \to \mathsf{Mat}_{n\times n}(\mathbb{C}) \otimes \mathsf{Mat}_{n\times n}(\mathbb{C})$ be a unitary solution of the associative Yang–Baxter equation (1.7) having a Laurent expansion of the form (1.9) and $\bar{r}_0(y_1, y_2)$ be the corresponding solution of the classical Yang–Baxter equation. If $\phi : (\mathbb{C}^2_{(v;y)}, 0) \to \mathsf{GL}_n(\mathbb{C})$ is the germ of a holomorphic function such that*

(1.10)

$$s(v_1, v_2; y_1, y_2) := \big(\phi(v_1; y_1) \otimes \phi(v_2; y_2)\big) r(v; y_1, y_2) \big(\phi(v_2; y_1)^{-1} \otimes \phi(v_1; y_2)^{-1}\big)$$

is again a function of $v = v_2 - v_1$, then we have

$$s(v; y_1, y_2) = \frac{1}{v}\mathbb{1}\otimes\mathbb{1} + s_0(y_1, y_2) + v s_1(y_1, y_2) + v^2 s_2(y_1, y_2) + \dots$$

and $\bar{r}_0(y_1, y_2)$ and $\bar{s}_0(y_1, y_2)$ are equivalent in the sense of the relation (1.2).

Proof. We denote $v = v_1 - v_2$ and $h = v_2$. Then $v_1 = v + h$ and using the Taylor expansion of ϕ with respect to v, we may rewrite (1.10) in the form

$$\begin{aligned}
&\left(\phi(h; y_1) + v\phi'(h; y_1) + \frac{v^2}{2}\phi''(h; y_1) + \dots\right) \otimes \phi(h; y_2)\cdot \\
&\qquad\qquad \cdot\left(\frac{\mathbb{1}\otimes\mathbb{1}}{v} + r_0(y_1, y_2) + v r_1(y_1, y_2) + \dots\right) \\
&= \left(\sum_{i\in\mathbb{Z}} s_i(y_1, y_2) v^i\right) \cdot \left(\phi(h; y_1) \otimes \left(\phi(h; y_2) + v\phi'(h; y_2) + \frac{v^2}{2}\phi''(h; y_2) + \dots\right)\right),
\end{aligned}$$