ROGER PENROSE

Author of *THE EMPEROR'S NEW MIND*

THE ROAD TO REALITY

A COMPLETE GUIDE TO THE LAWS OF THE UNIVERSE

"A comprehensive guide to physics' big picture, and to the thoughts of one of the world's most original thinkers." —*The New York Times*

NATIONAL BESTSELLER

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Preface

The purpose of this book is to convey to the reader some feeling for what is surely one of the most important and exciting voyages of discovery that humanity has embarked upon. This is the search for the underlying principles that govern the behaviour of our universe. It is a voyage that has lasted for more than two-and-a-half millennia, so it should not surprise us that substantial progress has at last been made. But this journey has proved to be a profoundly difficult one, and real understanding has, for the most part, come but slowly. This inherent difficulty has led us in many false directions; hence we should learn caution. Yet the 20th century has delivered us extraordinary new insights—some so impressive that many scientists of today have voiced the opinion that we may be close to a basic understanding of *all* the underlying principles of physics. In my descriptions of the current fundamental theories, the 20th century having now drawn to its close, I shall try to take a more sober view. Not all my opinions may be welcomed by these 'optimists', but I expect further changes of direction greater even than those of the last century.

The reader will find that in this book I have not shied away from presenting mathematical formulae, despite dire warnings of the severe reduction in readership that this will entail. I have thought seriously about this question, and have come to the conclusion that what I have to say cannot reasonably be conveyed without a certain amount of mathematical notation and the exploration of genuine mathematical concepts. The understanding that we have of the principles that actually underlie the behaviour of our physical world indeed depends upon some appreciation of its mathematics. Some people might take this as a cause for despair, as they will have formed the belief that they have no capacity for mathematics, no matter at how elementary a level. How could it be possible, they might well argue, for them to comprehend the research going on at the cutting edge of physical theory if they cannot even master the manipulation of *fractions*? Well, I certainly see the difficulty.

Yet I am an optimist in matters of conveying understanding. Perhaps I am an incurable optimist. I wonder whether those readers who cannot manipulate fractions—or those who claim that they cannot manipulate fractions—are not deluding themselves at least a little, and that a good proportion of them actually have a potential in this direction that they are not aware of. No doubt there are some who, when confronted with a line of mathematical symbols, however simply presented, can see only the stern face of a parent or teacher who tried to force into them a non-comprehending parrot-like apparent competence—a duty, and a duty alone—and no hint of the magic or beauty of the subject might be allowed to come through. Perhaps for some it is too late; but, as I say, I am an optimist and I believe that there are many out there, even among those who could never master the manipulation of fractions, who have the capacity to catch some glimpse of a wonderful world that I believe must be, to a significant degree, genuinely accessible to them.

One of my mother's closest friends, when she was a young girl, was among those who could not grasp fractions. This lady once told me so herself after she had retired from a successful career as a ballet dancer. I was still young, not yet fully launched in my activities as a mathematician, but was recognized as someone who enjoyed working in that subject. 'It's all that cancelling', she said to me, 'I could just never get the hang of cancelling.' She was an elegant and highly intelligent woman, and there is no doubt in my mind that the mental qualities that are required in comprehending the sophisticated choreography that is central to ballet are in no way inferior to those which must be brought to bear on a mathematical problem. So, grossly overestimating my expositional abilities, I attempted, as others had done before, to explain to her the simplicity and logical nature of the procedure of 'cancelling'.

I believe that my efforts were as unsuccessful as were those of others. (Incidentally, her father had been a prominent scientist, and a Fellow of the Royal Society, so she must have had a background adequate for the comprehension of scientific matters. Perhaps the 'stern face' could have been a factor here, I do not know.) But on reflection, I now wonder whether she, and many others like her, did not have a more rational hang-up—one that with all my mathematical glibness I had not noticed. There is, indeed, a profound issue that one comes up against again and again in mathematics and in mathematical physics, which one first encounters in the seemingly innocent operation of cancelling a common factor from the numerator and denominator of an ordinary numerical fraction.

Those for whom the action of cancelling has become second nature, because of repeated familiarity with such operations, may find themselves insensitive to a difficulty that actually lurks behind this seemingly simple procedure. Perhaps many of those who find cancelling mysterious are seeing a certain profound issue more deeply than those of us who press onwards in a cavalier way, seeming to ignore it. What issue is this? It concerns the very way in which mathematicians can provide an existence to their mathematical entities and how such entities may relate to physical reality.

I recall that when at school, at the age of about 11, I was somewhat taken aback when the teacher asked the class what a fraction (such as $\frac{3}{8}$) actually is! Various suggestions came forth concerning the dividing up of pieces of pie and the like, but these were rejected by the teacher on the (valid) grounds that they merely referred to imprecise physical situations to which the precise mathematical notion of a fraction was to be *applied*; they did not tell us what that clear-cut mathematical notion actually *is*. Other suggestions came forward, such as $\frac{3}{8}$ is 'something with a 3 at the top and an 8 at the bottom with a horizontal line in between' and I was distinctly surprised to find that the teacher seemed to be taking these suggestions seriously! I do not clearly recall how the matter was finally resolved, but with the hindsight gained from my much later experiences as a mathematics undergraduate, I guess my schoolteacher was making a brave attempt at telling us the definition of a fraction in terms of the ubiquitous mathematical notion of an *equivalence class*.

What is this notion? How can it be applied in the case of a fraction and tell us what a fraction actually is? Let us start with my classmate's 'something with a 3 at the top and an 8 on the bottom'. Basically, this is suggesting to us that a fraction is specified by an ordered pair of whole numbers, in this case the numbers 3 and 8. But we clearly cannot regard the fraction as *being* such an ordered pair because, for example, the fraction $\frac{6}{16}$ is the same number as the fraction $\frac{3}{8}$, whereas the pair (6, 16) is certainly not the same as the pair (3, 8). This is only an issue of cancelling; for we can write $\frac{6}{16}$ as $\frac{3\times 2}{8\times 2}$ and then cancel the 2 from the top and the bottom to get $\frac{3}{8}$. Why are we allowed to do this and thereby, in some sense, 'equate' the pair (6, 16) with the pair (3, 8)? The mathematician's answer—which may well sound like a cop-out—has the cancelling rule just built in to the definition of a fraction: a pair of whole numbers $(a \times n,\ b \times n)$ is deemed to represent the same fraction as the pair (a, b) whenever n is any non-zero whole number (and where we should not allow b to be zero either).

But even this does not tell us what a fraction is; it merely tells us something about the way in which we represent fractions. What *is* a fraction, then? According to the mathematician's "equivalence class" notion, the fraction $\frac{3}{8}$, for example, simply is the infinite collection of all pairs

$$(3, 8), (-3, -8), (6, 16), (-6, -16), (9, 24), (-9, -24), (12, 32), \ldots,$$

where each pair can be obtained from each of the other pairs in the list by repeated application of the above cancellation rule.* We also need definitions telling us how to add, subtract, and multiply such infinite collections of pairs of whole numbers, where the normal rules of algebra hold, and how to identify the whole numbers themselves as particular types of fraction.

This definition covers all that we mathematically need of fractions (such as $\frac{1}{2}$ being a number that, when added to itself, gives the number 1, etc.), and the operation of cancelling is, as we have seen, built into the definition. Yet it seems all very formal and we may indeed wonder whether it really captures the intuitive notion of what a fraction is. Although this ubiquitous equivalence class procedure, of which the above illustration is just a particular instance, is very powerful as a pure-mathematical tool for establishing consistency and mathematical existence, it can provide us with very top-heavy-looking entities. It hardly conveys to us the intuitive notion of what $\frac{3}{8}$ is, for example! No wonder my mother's friend was confused.

In my descriptions of mathematical notions, I shall try to avoid, as far as I can, the kind of mathematical pedantry that leads us to define a fraction in terms of an 'infinite class of pairs' even though it certainly has its value in mathematical rigour and precision. In my descriptions here I shall be more concerned with conveying the idea—and the beauty and the magic—inherent in many important mathematical notions. The idea of a fraction such as $\frac{3}{8}$ is simply that it is some kind of an entity which has the property that, when added to itself 8 times in all, gives 3. The magic is that the idea of a fraction actually works despite the fact that we do not really directly experience things in the physical world that are exactly quantified by fractions—pieces of pie leading only to approximations. (This is quite unlike the case of natural numbers, such as 1, 2, 3, which do precisely quantify numerous entities of our direct experience.) One way to see that fractions do make consistent sense is, indeed, to use the 'definition' in terms of infinite collections of pairs of integers (whole numbers), as indicated above. But that does not mean that $\frac{3}{8}$ actually *is* such a collection. It is better to think of $\frac{3}{8}$ as being an entity with some kind of (Platonic) existence of its own, and that the infinite collection of pairs is merely one way of our coming to terms with the consistency of this type of entity. With familiarity, we begin to believe that we can easily grasp a notion like $\frac{3}{8}$ as something that has its own kind of existence, and the idea of an 'infinite collection of pairs' is merely a pedantic device—a device that quickly recedes from our imaginations once we have grasped it. Much of mathematics is like that.

* This is called an 'equivalence class' because it actually is a class of entities (the entities, in this particular case, being pairs of whole numbers), each member of which is deemed to be equivalent, in a specified sense, to each of the other members.

To mathematicians (at least to most of them, as far as I can make out), mathematics is not just a cultural activity that we have ourselves created, but it has a life of its own, and much of it finds an amazing harmony with the physical universe. We cannot get any deep understanding of the laws that govern the physical world without entering the world of mathematics. In particular, the above notion of an equivalence class is relevant not only to a great deal of important (but confusing) mathematics, but a great deal of important (and confusing) physics as well, such as Einstein's general theory of relativity and the 'gauge theory' principles that describe the forces of Nature according to modern particle physics. In modern physics, one cannot avoid facing up to the subtleties of much sophisticated mathematics. It is for this reason that I have spent the first 16 chapters of this work directly on the description of mathematical ideas.

What words of advice can I give to the reader for coping with this? There are four different levels at which this book can be read. Perhaps you are a reader, at one end of the scale, who simply turns off whenever a mathematical formula presents itself (and some such readers may have difficulty with coming to terms with fractions). If so, I believe that there is still a good deal that you can gain from this book by simply skipping all the formulae and just reading the words. I guess this would be much like the way I sometimes used to browse through the chess magazines lying scattered in our home when I was growing up. Chess was a big part of the lives of my brothers and parents, but I took very little interest, except that I enjoyed reading about the exploits of those exceptional and often strange characters who devoted themselves to this game. I gained something from reading about the brilliance of moves that they frequently made, even though I did not understand them, and I made no attempt to follow through the notations for the various positions. Yet I found this to be an enjoyable and illuminating activity that could hold my attention. Likewise, I hope that the mathematical accounts I give here may convey something of interest even to some profoundly non-mathematical readers if they, through bravery or curiosity, choose to join me in my journey of investigation of the mathematical and physical ideas that appear to underlie our physical universe. Do not be afraid to skip equations (I do this frequently myself) and, if you wish, whole chapters or parts of chapters, when they begin to get a mite too turgid! There is a great variety in the difficulty and technicality of the material, and something elsewhere may be more to your liking. You may choose merely to dip in and browse. My hope is that the extensive cross-referencing may sufficiently illuminate unfamiliar notions, so it should be possible to track down needed concepts and notation by turning back to earlier unread sections for clarification.

At a second level, you may be a reader who is prepared to peruse mathematical formulae, whenever such is presented, but you may not have the inclination (or the time) to verify for yourself the assertions that I shall be making. The confirmations of many of these assertions constitute the solutions of the exercises that I have scattered about the mathematical portions of the book. I have indicated three levels of difficulty by the icons –

 very straight forward

 needs a bit of thought

 not to be undertaken lightly.

It is perfectly reasonable to take these on trust, if you wish, and there is no loss of continuity if you choose to take this position.

If, on the other hand, you are a reader who does wish to gain a facility with these various (important) mathematical notions, but for whom the ideas that I am describing are not all familiar, I hope that working through these exercises will provide a significant aid towards accumulating such skills. It is always the case, with mathematics, that a little direct experience of thinking over things on your own can provide a much deeper understanding than merely reading about them. (If you need the solutions, see the website www.roadsolutions.ox.ac.uk.)

Finally, perhaps you are already an expert, in which case you should have no difficulty with the mathematics (most of which will be very familiar to you) and you may have no wish to waste time with the exercises. Yet you may find that there is something to be gained from my own perspective on a number of topics, which are likely to be somewhat different (sometimes very different) from the usual ones. You may have some curiosity as to my opinions relating to a number of modern theories (e.g. supersymmetry, inflationary cosmology, the nature of the Big Bang, black holes, string theory or M-theory, loop variables in quantum gravity, twistor theory, and even the very foundations of quantum theory). No doubt you will find much to disagree with me on many of these topics. But controversy is an important part of the development of science, so I have no regrets about presenting views that may be taken to be partly at odds with some of the mainstream activities of modern theoretical physics.

It may be said that this book is really about the relation between mathematics and physics, and how the interplay between the two strongly influences those drives that underlie our searches for a better theory of the universe. In many modern developments, an essential ingredient of these drives comes from the judgement of mathematical beauty, depth, and sophistication. It is clear that such mathematical influences can be vitally important, as with some of the most impressively successful achievements of 20th-century physics: Dirac's equation for the electron, the general framework of quantum mechanics, and Einstein's general relativity. But in all these cases, physical considerations—ultimately observational ones—have provided the overriding criteria for acceptance. In many of the modern ideas for fundamentally advancing our understanding of the laws of the universe, adequate physical criteria—i.e. experimental data, or even the possibility of experimental investigation—are not available. Thus we may question whether the accessible mathematical desiderata are sufficient to enable us to estimate the chances of success of these ideas. The question is a delicate one, and I shall try to raise issues here that I do not believe have been sufficiently discussed elsewhere.

Although, in places, I shall present opinions that may be regarded as contentious, I have taken pains to make it clear to the reader when I am actually taking such liberties. Accordingly, this book may indeed be used as a genuine guide to the central ideas (and wonders) of modern physics. It is appropriate to use it in educational classes as an honest introduction to modern physics—as that subject is understood, as we move forward into the early years of the third millennium.

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Notation

(Not to be read until you are familiar with the concepts, but perhaps find the fonts confusing!)

I have tried to be reasonably consistent in the use of particular fonts in this book, but as not all of this is standard, it may be helpful to the reader to have the major usage that I have adopted made explicit.

Italic lightface (Greek or Latin) letters, such as in $w^2, p^n, \log z$, $\cos\theta$, $e^{i\theta}$, or e^x are used in the conventional way for mathematical variables which are numerical or scalar quantities; but established numerical constants, such as e, i, or π or established functions such as sin, cos, or log are denoted by upright letters. Standard physical constants such as c, G, h, $\hbar$, g, or k are italic, however.

A vector or tensor quantity, when being thought of in its (abstract) entirety, is denoted by a boldface italic letter, such as $\boldsymbol{R}$ for the Riemann curvature tensor, while its set of components might be written with lightface italic letters (both for the kernel symbol its indices) as R_{abcd}. In accordance with the abstract-index notation, introduced here in §12.8, the quantity R_{abcd} may alternatively stand for the entire tensor $\boldsymbol{R}$, if this interpretation is appropriate, and this should be clear from the text. Abstract linear transformations are kinds of tensors, and boldface italic letters such as $\boldsymbol{T}$ are used for such entities also. The abstract-index form $T^a{}_b$ is also used here for an abstract linear transformation, where appropriate, the staggering of the indices making clear the precise connection with the ordering of matrix multiplication. Thus, the (abstract-)index expression $S^a{}_bT^b{}_c$ stands for the product $\boldsymbol{ST}$ of linear transformations. As with general tensors, the symbols $S^a{}_b$ and $T^b{}_c$ could alternatively (according to context or explicit specification in the text) stand for the corresponding arrays of components—these being *matrices*—for which the corresponding bold upright letters **S** and **T** can also be used. In that case, **ST** denotes the corresponding matrix product. This 'ambivalent' interpretation of symbols such as R_{abcd} or $S^a{}_b$ (either standing for the array of components or for the abstract tensor itself) should not cause confusion, as the algebraic (or differential) relations that these symbols are subject to are identical for both interpretations. A third notation for such quantities—the *diagrammatic* notation—is also sometimes used here, and is described in Figs. 12.17, 12.18, 14.6, 14.7, 14.21, 19.1 and elsewhere in the book.

There are places in this book where I need to distinguish the 4-dimensional spacetime entities of relativity theory from the corresponding ordinary 3-dimensional purely spatial entities. Thus, while a boldface italic notation might be used, as above, such as $\boldsymbol{p}$ or $\boldsymbol{x}$, for the 4-momentum or 4-position, respectively, the corresponding 3-dimensional purely spatial entities would be denoted by the corresponding upright bold letters $\mathbf{p}$ or $\mathbf{x}$. By analogy with the notation $\mathbf{T}$ for a matrix, above, as opposed to $\boldsymbol{T}$ for an abstract linear transformation, the quantities $\mathbf{p}$ and $\mathbf{x}$ would tend to be thought of as 'standing for' the three spatial components, in each case, whereas $\boldsymbol{p}$ and $\boldsymbol{x}$ might be viewed as having a more abstract component-free interpretation (although I shall not be particularly strict about this). The Euclidean 'length' of a 3-vector quantity $\mathbf{a} = (a_1,a_2,a_3)$ may be written a, where $a^2 = a_1^2 + a_2^2 + a_3^2$, and the scalar product of $\mathbf{a}$ with $\mathbf{b} = (b_1,b_2,b_3)$, written $\mathbf{a} \bullet \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$. This 'dot' notation for scalar products applies also in the general n-dimensional context, for the scalar (or inner) product $\boldsymbol{\alpha} \bullet \boldsymbol{\xi}$ of an abstract covector $\boldsymbol{\alpha}$ with a vector $\boldsymbol{\xi}$.

A notational complication arises with quantum mechanics, however, since physical quantities, in that subject, tend to be represented as linear operators. I do not adopt what is a quite standard procedure in this context, of putting 'hats' (circumflexes) on the letters representing the quantum-operator versions of the familiar classical quantities, as I believe that this leads to an unnecessary cluttering of symbols. (Instead, I shall tend to adopt a philosophical standpoint that the classical and quantum entities are really the 'same'—and so it is fair to use the same symbols for each—except that in the classical case one is justified in ignoring quantities of the order of $\hbar$, so that the classical commutation properties $ab = ba$ can hold, whereas in quantum mechanics, ab might differ from ba by something of order $\hbar$.) For consistency with the above, such linear operators would seem to have to be denoted by italic bold letters (like $\boldsymbol{T}$), but that would nullify the philosophy and the distinctions called for in the preceding paragraph. Accordingly, with regard to specific quantities, such as the momentum $\mathbf{p}$ or $\boldsymbol{p}$, or the position $\mathbf{x}$ or $\boldsymbol{x}$, I shall tend to use the same notation as in the classical case, in line with what has been said earlier in this paragraph. But for less specific quantum operators, bold italic letters such as $\boldsymbol{Q}$ will tend to be used.

The shell letters $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{F}_q$, respectively, for the system of natural numbers (i.e. non-negative integers), integers, real numbers, complex numbers, and the finite field with q elements (q being some power of a prime number, see §16.1), are now standard in mathematics, as are the corresponding $\mathbb{N}^n$, $\mathbb{Z}^n$, $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{F}_q^n$, for the systems of ordered n-tuples of such numbers. These are canonical mathematical entities in standard use. In this book (as is not all that uncommon), this notation is extended to some other standard mathematical structures such as Euclidean 3-space $\mathbb{E}^3$ or, more generally, Euclidean n-space $\mathbb{E}^n$. In frequent use in this book is the standard flat 4-dimensional Minkowski spacetime, which is itself a kind of 'pseudo-' Euclidean space, so I use the shell letter $\mathbb{M}$ for this space (with $\mathbb{M}^n$ to denote the n-dimensional version—a 'Lorentzian' spacetime with 1 time and $(n-1)$ space dimensions). Sometimes I use $\mathbb{C}$ as an adjective, to denote 'complexified', so that we might consider the complex Euclidean 4-space, for example, denoted by $\mathbb{CE}^n$. The shell letter $\mathbb{P}$ can also be used as an adjective, to denote 'projective' (see §15.6), or as a noun, with $\mathbb{P}^n$ denoting projective n-space (or I use $\mathbb{RP}^n$ or $\mathbb{CP}^n$ if it is to be made clear that we are concerned with real or complex projective n-space, respectively). In twistor theory (Chapter 33), there is the complex 4-space $\mathbb{T}$, which is related to $\mathbb{M}$ (or its complexification $\mathbb{CM}$) in a canonical way, and there is also the projective version $\mathbb{PT}$. In this theory, there is also a space $\mathbb{N}$ of *null* twistors (the double duty that this letter serves causing no conflict here), and its projective version $\mathbb{PN}$.

The adjectival role of the shell letter $\mathbb{C}$ should not be confused with that of the lightface sans serif C, which here stands for 'complex conjugate of' (as used in §13.1,2). This is basically similar to another use of C in particle physics, namely *charge conjugation*, which is the operation which interchanges each particle with its antiparticle (see Chapters 25, 30). This operation is usually considered in conjunction with two other basic particle-physics operations, namely P for *parity* which refers to the operation of reflection in a mirror, and T, which refers to *time-reversal*. Sans serif letters which are bold serve a different purpose here, labelling *vector spaces*, the letters **V**, **W**, and **H**, being most frequently used for this purpose. The use of **H**, is specific to the Hilbert spaces of quantum mechanics, and $\mathbf{H}^n$ would stand for a Hilbert space of n complex dimensions. Vector spaces are, in a clear sense, flat. Spaces which are (or could be) *curved* are denoted by script letters, such as $\mathcal{M}$, $\mathcal{S}$, or $\mathcal{T}$, where there is a special use for the particular script font $\mathscr{I}$ to denote *null infinity*. In addition, I follow a fairly common convention to use script letters for Lagrangians ($\mathcal{L}$) and Hamiltonians ($\mathcal{H}$), in view of their very special status in physical theory.

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