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Differential Equations in Abstract Spaces

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Preface

The theory of differential equations in abstract spaces is a fascinating field with important applications to a number of areas of analysis and other branches of mathematics. At the present time, there is no single book that is self-contained and simple enough to appeal to the beginner. Furthermore, if one desires to give a course so as to expose the student to this branch of research, such a book becomes handy. This being the motivation, the aim of our book is, in fact, to introduce the nonspecialist to this elegant theory and powerful techniques. But for some familiarity with the elements of functional analysis, all the important results used in this book are carefully stated in the appendixes so that, for the most part, no other references are needed. The required theory, from the calculus of abstract functions and the theory of semigroups of operators, used in connection with differential equations in Banach spaces is treated in detail.

We have tried to present the fundamental theory of differential equations in Banach spaces: the first three chapters form an integrated whole together with, perhaps, Sections 6.1 and 6.3 of Chapter 6. Chapter 4 is devoted to the study of differential inequalities, mostly, in Hilbert spaces. The theory developed in Chapter 5 is interesting in itself and could be read independently. This also applies to Chapter 4. Throughout the book we give a number of examples and applications to functional and partial differential equations which help to illustrate the abstract results developed. In most sections there are several problems with hints directly related to the material in the text. The notes at the end of each chapter indicate the sources which have been consulted and those whose ideas are developed. Several references are also included for further study on the subject. We hope that the reader who is familiar with the contents of this book will be fully equipped to contribute to this field as well as read with ease the current literature.

Acknowledgments

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Contents

Chapter 1

The Calculus of Abstract Functions

1.0. Introduction

In this preliminary chapter the reader will be familiarized with those parts of the calculus of abstract functions that are essential in the study of differential equations in Banach and Hilbert spaces. By an abstract function we mean a function mapping an interval of the real line into a Banach space. We begin by defining weak and strong continuity and differentiability of abstract functions and prove a form of the mean value theorem for abstract functions. Next we develop the Riemann integral for abstract functions and those properties of this integral which are constantly used in the text. We then outline abstract integrals of the Lebesgue type (Pettis and Bochner integrals) and state some basic results. We also sketch the abstract Stieltjes integral for functions mapping a Banach space into another Banach space. Finally we treat in some detail the Gateaux and Fréchet differential of functions mapping a Banach space into another Banach space.

1.1. Abstract Functions

Let X be a Banach space over the field of real numbers and for any $x \in X$, let $\|x\|$ denote the norm of x. Let J be any interval of the real line R. A function $x: J \to X$ is called an *abstract* function. A function $x(t)$ is said to be *continuous* at the point $t_0 \in J$, if $\lim_{t \to t_0} \|x(t) - x(t_0)\| = 0$; if $x: J \to X$ is continuous at each point of J, then we say that x is continuous on J and we write $x \in C[J, X]$.

Abstract functions are in many ways reminiscent of ordinary functions. For example, a continuous abstract function maps compact sets into compact sets. Also, a continuous abstract function on a compact set is uniformly continuous. These statements can be proved in the same way that we prove them in a metric space.

An abstract function $x(t)$ is said to be

(i) *Lipschitz continuous* on J with Lipschitz constant K if

$$\|x(t_1) - x(t_2)\| \leqslant K|t_1 - t_2|, \qquad t_1, t_2 \in J;$$

(ii) *uniformly Hölder continuous* on J with Hölder constant K and exponent β, if

$$\|x(t_1) - x(t_2)\| \leqslant K|t_1 - t_2|^{\beta}, \qquad t_1, t_2 \in J, \quad 0 < \beta \leqslant 1.$$

It is clear that Lipschitz continuity implies Hölder continuity (with $\beta = 1$) but the converse fails as the classical example $x(t) = \sqrt{t}$, $K = 1$, $\beta = \frac{1}{2}$, shows. The (strong) derivative of $x(t)$ is defined by

$$x'(t) = \lim_{\Delta t \to 0} [x(t + \Delta t) - x(t)]/\Delta t$$

where the limit is taken in the strong sense, that is,

$$\lim_{\Delta t \to 0} \|[x(t+h) - x(t)]/h - x'(t)\| = 0.$$

The foregoing concepts of continuity and differentiability are defined in the *strong* sense. The corresponding *weak* concepts are defined as follows. Let X^* denote the conjugate of X, that is, the space of all bounded linear functionals on X. An abstract function $x(t)$ is said to be *weakly continuous* (*weakly differentiable*) at $t = t_0$ if for every $\phi \in X^*$, the scalar function $\phi[x(t)]$ is continuous (differentiable) at $t = t_0$. In the sequel, limits shall be understood in the strong sense unless we write w-lim to indicate that we are taking the weak limit. Also continuity and differentiability shall be understood in the strong sense unless otherwise specified.

A family $F = \{x(t)\}$ of abstract functions with domain $[a, b]$ and range in X is said to be *equicontinuous* if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon)$ which depends only on ε such that for every $t_1, t_2 \in [a, b]$ with $|t_1 - t_2| < \delta$, $|x(t_1) - x(t_2)| < \varepsilon$ for all $x \in F$.

The following form of the Ascoli–Arzela theorem for abstract functions will be useful. Its proof is a special case of a more general theorem [63].

THEOREM 1.1.1. Let $F = \{x(t)\}$ be an equicontinuous family of functions from $[a, b]$ into X. Let $\{x_n(t)\}_{n=1}^{\infty}$ be a sequence in F such that for each $t_1 \in [a, b]$ the set $\{x_n(t_1) : n \geqslant 1\}$ is relatively compact in X. Then, there is a subsequence $\{x_{n_k}(t)\}_{k=1}^{\infty}$ which is uniformly convergent on $[a, b]$.

1.2. The Mean Value Theorem

For real-valued functions $x(t)$, the mean value theorem is written as an equality

$$x(b) - x(a) = x'(c)(b-a), \qquad a < c < b.$$

There is nothing similar to it as soon as $x(t)$ is a vector-valued function as one can see from the example $x(t) = (-1 + \cos t, \sin t)$ with $a = 0$ and $b = 2\pi$.

For abstract functions the following form of the mean value theorem is useful.

THEOREM 1.2.1. If $x \in C[[a, b], X]$ and $\|x'(t)\| \leqslant K$, $a < t < b$, then

$$\|x(b) - x(a)\| \leqslant K(b-a).$$

Proof: Choose a functional $\phi \in X^*$ such that $\|\phi\| = 1$ and $\phi[x(b) - x(a)] = \|x(b) - x(a)\|$. Such a choice of ϕ is possible in view of Appendix III. Define the real-valued function $f(t) = \phi[x(t)]$. Then

$$[f(t+h) - f(t)]/h = \phi[x(t+h) - x(t)]/h.$$

Since ϕ is a continuous linear functional and $x'(t)$ exists, it follows that $f'(t)$ exists for $a < t < b$ and $f'(t) = \phi[x'(t)]$. Hence, the classical mean value theorem applies to $f(t)$ and consequently there exists a τ, such that

$$f(b) - f(a) = f'(\tau)(b-a), \qquad a < \tau < b. \tag{1.2.1}$$

In view of (1.2.1) and the choice of ϕ we obtain

$$\begin{aligned}\|x(b)-x(a)\| &= \phi[x(b)-x(a)] = f'(\tau)(b-a) = \phi[x'(\tau)](b-a) \\ &\leqslant \|x'(\tau)\|(b-a) \leqslant K(b-a)\end{aligned}$$

and the proof is complete.

COROLLARY 1.2.1. If $x \in C[[a,b],X]$ and $x'(t)=0$, $a<t<b$, then $x(t) \equiv \text{const}$.

PROBLEM 1.2.1. Let $x \in C[[a,b],X]$ and $f \in C[[a,b],R]$. Assume that x and f have derivatives on $[a,b]-D$ where D is a denumerable set and $\|x'(t)\| \leqslant f'(t)$, $t \in [a,b]-D$. Then

$$\|x(b)-x(a)\| \leqslant f(b)-f(a).$$

1.3. The Riemann Integral for Abstract Functions

Here we shall define the Riemann integral for abstract functions and prove the fundamental theorem of calculus. We also define improper integrals and discuss some properties which will be constantly used in this book.

Let $x:[a,b] \to X$ be an abstract function. We denote the partition $(a=t_0<t_1<\cdots<t_n=b)$ together with the points τ_i $(t_i \leqslant \tau_i \leqslant t_{i+1}$, $i=0,1,2,\ldots,n-1)$ by π and set $|\pi| = \max_i |t_{i+1}-t_i|$. We form the Riemann sum

$$S_\pi = \sum_{i=0}^{n-1} (t_{i+1}-t_i)\, x(\tau_i).$$

If $\lim S_\pi$ exists as $|\pi| \to 0$ and defines an element I in X which is independent of π, then I is called the *Riemann integral* of the function $x(t)$ and is denoted by

$$I = \int_a^b x(t)\, dt.$$

THEOREM 1.3.1. If $x \in C[[a,b],X]$, then the Riemann integral $\int_a^b x(t)\, dt$ exists.

The proof of this theorem makes use of the facts that a continuous function on a closed, bounded interval is uniformly continuous and that X is complete. We shall omit the proof.

Using the definition of Riemann integral one can easily verify the following properties:

(i) $$\int_a^b x(t)\,dt = -\int_b^a x(t)\,dt$$

provided that one of the integrals exist.

(ii) $$\int_a^b x(t)\,dt = \int_a^c x(t)\,dt + \int_c^b x(t)\,dt, \qquad a < c < b$$

provided that the integral on the left exists.

(iii) If $x(t) \equiv x_0$ for all $t \in [a, b]$, then

$$\int_a^b x_0\,dt = (b-a)\,x_0.$$

(iv) If $t = \omega(\tau)$ is an increasing continuous function on $[\alpha, \beta]$ with $a = \omega(\alpha)$ and $b = \omega(\beta)$, then

$$\int_a^b x(t)\,dt = \int_\alpha^\beta x[\omega(\tau)]\,\omega'(\tau)\,d\tau$$

provided that the integral on the left exists.

(v) If $x \in C[[a, b], X]$, then

$$\left\| \int_a^b x(t)\,dt \right\| \leqslant \int_a^b \|x(t)\|\,dt.$$

Indeed, from the definition of the Riemann sum we have

$$\begin{aligned} \|S_\pi\| &\leqslant \left\| \sum_{i=0}^{n-1} (t_{i+1}-t_i)\,x(\tau_i) \right\| \\ &\leqslant \sum_{i=0}^{n-1} (t_{i+1}-t_i)\,\|x(\tau_i)\| \end{aligned}$$

and the result follows by taking limits as $|\pi| \to 0$ and the fact that $\|x(t)\|$ is continuous and hence integrable on $[a, b]$.

THEOREM 1.3.2. If $\{x_n(t)\}$ is a sequence of continuous abstract functions which converges uniformly to a necessarily continuous, abstract function $x(t)$ on the interval $[a, b]$, then

$$\lim_{n\to\infty} \int_a^b x_n(t)\,dt = \int_a^b x(t)\,dt.$$

Proof: We have

$$\left\| \int_a^b x_n(t)\,dt - \int_a^b x(t)\,dt \right\| \leqslant \int_a^b \|x_n(t) - x(t)\|\,dt$$
$$\leqslant \max_{[a,b]} \|x_n(t) - x(t)\|\,(b-a) \to 0 \qquad \text{as } n \to \infty,$$

which proves the stated result.

THEOREM 1.3.3. If $x \in C[[a,b], X]$, then

$$(d/dt) \int_a^t x(s)\,ds = x(t), \qquad a \leqslant t \leqslant b.$$

Proof: Set $y(t) = \int_a^t x(s)\,ds$. Then, in view of the fact that $x(t)$ is uniformly continuous on $[a,b]$, we have

$$\|[y(t+h) - y(t)]/h - x(t)\| = \left\| h^{-1} \int_t^{t+h} [x(s) - x(t)]\,ds \right\|$$
$$\leqslant \max_{|s-t| \leqslant |h|} \|x(s) - x(t)\| \to 0 \qquad \text{as} \quad h \to 0$$

and the proof is complete.

THEOREM 1.3.4. If the function $x\colon [a,b] \to X$ is continuously differentiable on (a,b), then for any $\alpha, \beta \in (a,b)$ the following formula is true:

$$\int_\alpha^\beta x'(s)\,ds = x(\beta) - x(\alpha).$$

Proof: By Theorem 1.3.3

$$(d/dt)\left[\int_\alpha^t x'(s)\,ds - x(t)\right] \equiv 0, \qquad \alpha \leqslant t \leqslant \beta.$$

Hence

$$\int_\alpha^t x'(s)\,ds - x(t) = \text{const.} \tag{1.3.1}$$

For $t = \alpha$ we find the value of the const $= -x(\alpha)$ and the result follows by setting $t = \beta$ in (1.3.1).

REMARK 1.3.1. In elementary calculus, if x is continuous on $[a,b]$, then

$$\int_a^b x(t)\,dt = (b-a)\,x(\xi)$$

for some $\xi \in (a, b)$. This is not true for vector-valued continuous functions x as we can see from the simple example $x(t) = (\cos t, \sin t)$, $a = 0$ and $b = \pi$.

Let x: $[a, b) \to X$ be an abstract function which is not defined at $b \leqslant \infty$. The *improper integral* $\int_a^b x(t)\,dt$ is defined as

$$\lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} x(t)\,dt \qquad \text{if} \quad b < \infty$$

and as

$$\lim_{M \to \infty} \int_a^{M} x(t)\,dt \qquad \text{if} \quad b = \infty$$

provided that the limit exists.

The following theorem which asserts that integration commutes with closed operators (in particular, integration commutes with bounded operators) will be used often.

THEOREM 1.3.5. Let A on $D(A)$ be a closed operator in the Banach space X and $x \in C[[a, b), X]$ with $b \leqslant \infty$. Suppose that $x(t) \in D(A)$, $Ax(t)$ is continuous on $[a, b)$ and that the improper integrals

$$\int_a^b x(t)\,dt \qquad \text{and} \qquad \int_a^b Ax(t)\,dt$$

exist. Then

$$\int_a^b x(t)\,dt \in D(A) \qquad \text{and} \qquad A\int_a^b x(t)\,dt = \int_a^b Ax(t)\,dt.$$

Proof: We shall prove the theorem when $b < \infty$. The case $b = \infty$ is left to the reader. Set $c = b - \varepsilon$ where $\varepsilon > 0$ is sufficiently small. For a partition π of $[a, b]$ we have

$$f_n \equiv \sum_{i=0}^{n-1} x(\tau_i)(t_{i+1} - t_i) \in D(A)$$

and

$$g_n \equiv \sum_{i=0}^{n-1} Ax(\tau_i)(t_{i+1} - t_i) = Af_n.$$

In view of the hypotheses, as $n \to \infty$ and $|\pi| \to 0$

$$f_n \to \int_a^c x(t)\,dt \qquad \text{and} \qquad Af_n \to \int_a^c Ax(t)\,dt.$$

Since A is a closed operator on $D(A)$, it follows that

$$\int_a^c x(t)\,dt \in D(A) \qquad \text{and} \qquad A\int_a^c x(t)\,dt = \int_a^c Ax(t)\,dt.$$

Setting $c = b - n^{-1}$ in the previous result and using the definition of an improper integral and the fact that A is closed, the desired result follows upon taking limits as $n \to \infty$.

PROBLEM 1.3.1. Define the rectangle

$$R_0 = \{(t,x) \in R \times X\colon |t-t_0| \leqslant \alpha, \quad \|x-x_0\| \leqslant \beta\}.$$

Let $f\colon R_0 \to X$ be a function continuous in t for each fixed x

$$\|f(t,x)\| \leqslant M, \qquad (t,x) \in R_0$$

and

$$\|f(t,x_1) - f(t,x_2)\| \leqslant K\|x_1 - x_2\|, \qquad (t,x_1),(t,x_2) \in R_0.$$

Let α, β, K, M be positive constants such that $\alpha M \leqslant \beta$. Then there exists one and only one (strongly) continuously differentiable function $x(t)$ such that

$$dx(t)/dt = f[t,x(t)], \qquad |t-t_0| \leqslant \alpha \qquad \text{and} \qquad x(t_0) = x_0.$$

[Hint: Use Theorems 1.3.3 and 1.3.4 and the successive approximations

$$x_0(t) = x_0, \qquad x_n(t) = x_0 + \int_{t_0}^t f[s, x_{n-1}(s)]\,ds, \qquad |t-t_0| \leqslant \alpha.$$

Justify passing to the limit under the integral sign.]

1.4. Abstract Lebesgue Integrals

Here we shall outline the Bochner and Pettis integrals which are defined relative to the strong and weak topology, respectively, on a Banach space X. These integrals are of the Lebesgue type. Let us begin with some notions.

Let (Ω, S, m) be a measure space. The function $x\colon \Omega \to X$ is said to be

(i) *countably valued* in Ω if it assumes at most a countable set of values in X, assuming each value different from zero on a measurable set;

(ii) *weakly measurable* in Ω if the scalar function $\phi[x(\sigma)]$ is measurable for every $\phi \in X^*$;