Undergraduate Texts in Mathematics

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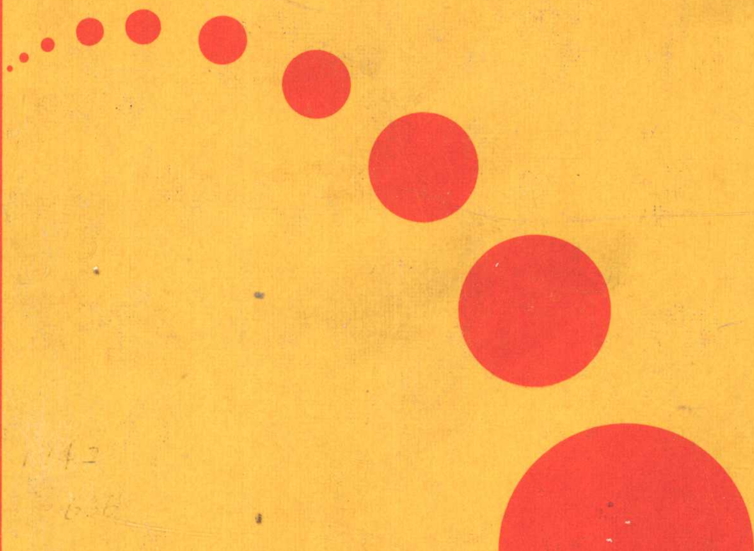
A First Course in Real Analysis



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Preface

The first course in analysis which follows elementary calculus is a critical one for students who are seriously interested in mathematics. Traditional advanced calculus was precisely what its name indicates—a course with topics in calculus emphasizing problem solving rather than theory. As a result students were often given a misleading impression of what mathematics is all about; on the other hand the current approach, with its emphasis on theory, gives the student insight in the fundamentals of analysis.

In *A First Course in Real Analysis* we present a theoretical basis of analysis which is suitable for students who have just completed a course in elementary calculus. Since the sixteen chapters contain more than enough analysis for a one year course, the instructor teaching a one or two quarter or a one semester junior level course should easily find those topics which he or she thinks students should have.

The first Chapter, on the real number system, serves two purposes. Because most students entering this course have had no experience in devising proofs of theorems, it provides an opportunity to develop facility in theorem proving. Although the elementary processes of numbers are familiar to most students, greater understanding of these processes is acquired by those who work the problems in Chapter 1. As a second purpose, we provide, for those instructors who wish to give a comprehensive course in analysis, a fairly complete treatment of the real number system including a section on mathematical induction.

Although Chapter 1 is useful as an introduction to analysis, the instructor of a short course may choose to begin with the second Chapter. Chapters 2 through 5 cover the basic theory of elementary calculus. Here we prove many of the theorems which are "stated without proof" in the standard freshman calculus course.

Crucial to the development of an understanding of analysis is the concept of a metric space. We discuss the fundamental properties of metric spaces in Chapter 6. Here we show that the notion of compactness is central and we prove several important results (including the Heine–Borel theorem) which are useful later on. The power of the general theory of metric spaces is aptly illustrated in Chapter 13, where we give the theory of contraction mappings and an application to differential equations. The study of metric spaces is resumed in Chapter 15, where the properties of functions on metric spaces are established. The student will also find useful in later courses results such as the Tietze extension theorem and the Stone-Weierstrass theorem, which are proved in detail.

Chapters 7, 8, and 12 continue the theory of differentiation and integration begun in Chapters 4 and 5. In Chapters 7 and 8, the theory of differentiation and integration in $\mathbb{R}_N$ is developed. Since the primary results for $\mathbb{R}_1$ are given in Chapters 4 and 5 only modest changes were necessary to prove the corresponding theorems in $\mathbb{R}_N$. In Chapter 12 we define the Riemann–Stieltjes integral and develop its principal properties.

Infinite sequences and series are the topics of Chapters 9 and 10. Besides subjects such as uniform convergence and power series, we provide in Section 9.5 a unified treatment of absolute convergence of multiple series. Here, in a discussion of unordered sums, we show that a separate treatment of the various kinds of summation of multiple series is entirely unnecessary. Chapter 10 on Fourier series contains a proof of the Dini test for convergence and the customary theorems on term-by-term differentiation and integration of such series.

In Chapter 14 we prove the Implicit Function theorem, first for a single equation and then for a system. In addition we give a detailed proof of the Lagrange multiplier rule, which is frequently stated but rarely proved. For completeness we give the details of the proof of the theorem on the change of variables in a multiple integral. Since the argument here is rather intricate, the instructor may wish to assign this section as optional reading for the best students.

Proofs of Green's and Stokes' theorems and the divergence theorem in $\mathbb{R}_2$ and $\mathbb{R}_3$ are given in Chapter 16. The methods used here are easily extended to the corresponding results in $\mathbb{R}_N$.

This book is also useful in freshman honors courses. It has been our experience that honors courses in freshman calculus frequently falter because it is not clear whether the honors student should work hard problems while he learns the regular calculus topics or should omit the regular topics entirely and concentrate on the underlying theory. In the first alternative, the honors student is hardly better off than the regular student taking the ordinary calculus course, while in the second the honors student fails to learn the *simple* problem solving techniques which, in fact, are useful later on. We believe that this dilemma can be resolved by employing two texts—one a standard calculus text and the other a book such as this one which provides the theoretical basis of the calculus in one and several dimensions. In this way the honors student gets both theory and practice. Chapters 2 through 5 and Chapters 7 and 8 provide a thorough account of the theory of elementary calculus which, along with a standard calculus book, is suitable as text material for a first year honors program.

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Berkeley, January 1977

Graduate Texts in Mathematics

Volume 12
Advanced Mathematical Analysis
by Richard Beals
1973. xi, 230p. paper/cloth.

Advanced Mathematical Analysis is a working introduction to real analysis, complex analysis, and functional analysis for advanced undergraduate and beginning graduate students.

Contents: Basic Concepts; Continuous Functions; Periodic Functions and Periodic Distributions; Hilbert Space and Fourier Series; Applications of Fourier Series; Complex Analysis; The Laplace Transform.

Volume 25
Real and Abstract Analysis
by E. Hewitt and K. Stromberg
3rd printing. 1975. viii, 476p. cloth.

Real and Abstract Analysis is designed for the standard graduate course in the theory of functions of real variables. All topics essential for the training of analysts are included. Special emphasis is on examples and applications from classical analysis.

Contents: Set Theory and Algebra; Topology and Continuous Functions; The Lebesque Integral; Function Spaces and Banach Spaces; Differentiation; Integration on Product Spaces.

Undergraduate Texts in Mathematics

APOSTOL. Introduction to Analytic Number Theory. xii, 338 pages. 1976.

CHUNG. Elementary Probability Theory with Stochastic Processes. 2nd printing. x, 325 pages. 1975.

FLEMING. Functions of Several Variables. 2nd ed. approx. 360 pages. 1977.

GRUENBERG/WEIR. Linear Geometry. 2nd ed. approx. 200 pages. 1977.

HALMOS. Finite-Dimensional Vector Spaces. 2nd ed. viii, 200 pages. 1974.

HALMOS. Naive Set Theory. vii, 104 pages. 1974.

KEMENY/SNELL. Finite Markov Chains. ix, 210 pages. 1976.

LAX/BURSTEIN/LAX. Calculus with Applications and Computing. Volume 1. xi, 513 pages. 1976.

SIGLER. Algebra. xii, 419 pages. 1976.

SINGER/THORPE. Lecture Notes on Elementary Topology and Geometry. viii, 232 pages. 1976.

Contents

The real number system 1

1.1 Axioms for a field

In an elementary calculus course the student learns the techniques of differentiation and integration and the skills needed for solving a variety of problems which use the processes of calculus. Most often, the principal theorems upon which calculus is based are stated without proof, while some of the auxiliary theorems are established in detail. To compensate for the missing proofs, most texts present arguments which show that the basic theorems are plausible. Frequently, a remark is added to the effect that rigorous proofs of these theorems can be found in advanced texts in analysis.

In this and the next four chapters we shall give a reasonably rigorous foundation to the processes of the calculus of functions of one variable. Calculus depends on the properties of the real number system. Therefore, to give a complete foundation for calculus, we would have to develop the real number system from the beginning. Since such a development is lengthy and would divert us from our aim of presenting a course in *analysis*, we shall assume the reader is familiar with the usual properties of the system of real numbers.

In this section we present a set of axioms that forms a logical basis for those processes of elementary algebra upon which calculus is based. Any collection of objects satisfying the axioms stated below is called a *field*. In particular the system of real numbers satisfies the field axioms, and we shall indicate how the customary laws of algebra concerning addition, subtraction, multiplication, and division follow from these axioms.

A thorough treatment would require complete proofs of all the theorems. In this section and the next we establish some of the elementary laws of algebra, and we refer the reader to a course in higher algebra where complete proofs of most of the theorems may be found. Since the reader is familiar with the laws of algebra and their use, we shall assume their validity throughout the remainder of the text. In addition, we shall suppose the reader is familiar with many facts about finite sets, positive integers, and so forth.

Throughout the book, we use the word *equals* or its symbol $=$ to stand for the words "is the same as." The reader should compare this with other uses for the symbol $=$ such as that in plane geometry where, for example, two line segments are said to be equal if they have the same length.

Axioms of addition and subtraction

A-1. Closure property. *If a and b are numbers, there is one and only one number, denoted $a + b$, called their **sum**.*

A-2. Commutative law. *For any two numbers a and b, the equality*

$$b + a = a + b$$

holds.

A-3. Associative law. *For all numbers a, b, and c, the equality*

$$(a + b) + c = a + (b + c)$$

holds.

A-4. Existence of a zero. *There is one and only one number 0, called zero, such that $a + 0 = a$ for any number a.*

It is not necessary to assume in Axiom A-4 that there is *only one* number 0 with the given property. The uniqueness of this number is easily established. Suppose 0 and $0'$ are two numbers such that $a + 0 = a$ and $a + 0' = a$ for *every* number a. Then $0 + 0' = 0$ and $0' + 0 = 0'$. By Axiom A-2, we have $0 + 0' = 0' + 0$ and so $0 = 0'$. The two numbers are the same.

A-5. Existence of a negative. *If a is any number, there is one and only one number x such that $a + x = 0$. This number is called the **negative of** a and is denoted by $-a$.*

As in Axiom A-4, it is not necessary to assume in Axiom A-5 that there is *only one* such number with the given property. The argument which establishes the uniqueness of the negative is similar to the one given after Axiom A-4.

Theorem 1.1. *If a and b are any numbers, then there is one and only one number x such that $a + x = b$. This number x is given by $x = b + (-a)$.*

Proof. We must establish two results: (i) that $b+(-a)$ satisfies the equation $a+x=b$ and (ii) that no other number satisfies this equation. To prove (i), suppose that $x=b+(-a)$. Then, using Axioms A-2 through A-4 we see that

$$a+x=a+[b+(-a)]=a+[(-a)+b]=[a+(-a)]+b=0+b=b.$$

Therefore (i) holds. To prove (ii), suppose that x is some number such that $a+x=b$. Adding $(-a)$ to both sides of this equation, we find that

$$(a+x)+(-a)=b+(-a).$$

Now,

$$\begin{aligned}(a+x)+(-a)&=a+[x+(-a)]=a+[(-a)+x]\\&=[a+(-a)]+x=0+x=x.\end{aligned}$$

We conclude that $x=b+(-a)$, and the uniqueness of the solution is established. □

Notation. The number $b+(-a)$ is denoted by $b-a$.

Thus far addition has been defined *only for two numbers*. By means of the associative law we can define addition for three, four and, in fact, any finite number of elements. Since $(a+b)+c$ and $a+(b+c)$ are the same, we define $a+b+c$ as this common value. The following lemma is an easy consequence of the associative and commutative laws of addition.

Lemma 1.1. *If a, b, and c are any numbers, then*

$$a+b+c=a+c+b=b+a+c=b+c+a=c+a+b=c+b+a.$$

The formal details of writing out a proof are left to the reader.
The next lemma is useful in the proof of Theorem 1.2 below.

Lemma 1.2. *If a, b, c, and d are numbers, then*

$$(a+c)+(b+d)=(a+b)+(c+d).$$

Proof. Using Lemma 1.1 and the axioms, we have

$$\begin{aligned}(a+c)+(b+d)&=[(a+c)+b]+d\\&=(a+c+b)+d=(a+b+c)+d\\&=[(a+b)+c]+d=(a+b)+(c+d).\end{aligned}$$ □

Theorem 1.2

(i) *If a is a number, then* $-(-a)=a$.
(ii) *If a and b are numbers, then*

$$-(a+b)=(-a)+(-b).$$

Proof. (i) From the definition of negative, we have

$$(-a)+[-(-a)]=0, \qquad (-a)+a=a+(-a)=0.$$

Axiom A-5 states that the negative of $(-a)$ is *unique*. Therefore, $a = -(-a)$. To establish (ii), we know from the definition of negative that

$$(a+b)+[-(a+b)]=0.$$

Furthermore, using Lemma 1.2, we have

$$(a+b)+[(-a)+(-b)]=[a+(-a)]+[b+(-b)]=0+0=0.$$

The result follows from the "only one" part of Axiom A-5. □

Theorem 1.2 can be stated in the familiar form: (i) *The negative of* $(-a)$ *is* a, *and* (ii) *The negative of a sum is the sum of the negatives.*

Axioms of multiplication and division.

M-1. Closure property. *If* a *and* b *are numbers, there is one and only one number, denoted by* ab (*or* $a \times b$ *or* $a \cdot b$), *called their* ***product***.

M-2. Commutative law. *For every two numbers* a *and* b, *the equality*

$$ba = ab$$

holds.

M-3. Associative law. *For all numbers* a, b, *and* c, *the equality*

$$(ab)c = a(bc)$$

holds.

M-4. Existence of a unit. *There is one and only one number* u, *different from zero, such that* $au = a$ *for every number* a. *This number* u *is called the* ***unit*** *and* (*as is customary*) *is denoted by* 1.

M-5. Existence of a reciprocal. *For each number* a *different from zero there is one and only one number* x *such that* $ax = 1$. *This number* x *is called the* ***reciprocal*** *of* a (*or the inverse of* a) *and is denoted by* a^{-1} (*or* $1/a$).

Remarks. Axioms M-1 through M-4 are the parallels of Axioms A-1 through A-4 with addition replaced by multiplication. However, M-5 is not the exact analogue of A-5, since the additional condition $a \neq 0$ is required. The reason for this is given below in Theorem 1.3, where it is shown that the result of multiplication of any number by zero is zero. In familiar terms we say that division by zero is excluded.