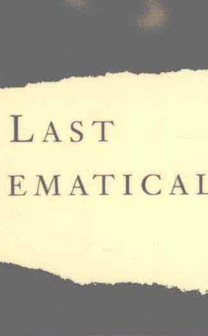
Is It Solved?

The World's Most Famous Math Problem

[*The* PROOF *of* FERMAT'S LAST THEOREM] *and* [OTHER MATHEMATICAL MYSTERIES]

ARILYN

OS SAVANT



The World's Most Famous Math Problem

[THE PROOF OF FERMAT'S LAST THEOREM AND OTHER MATHEMATICAL MYSTERIES]

Marilyn vos Savant

ST. MARTIN'S PRESS NEW YORK

For permission to reprint copyrighted material, grateful acknowledgement is made to the following sources:

The American Association for the Advancement of Science: Excerpts from *Science,* Volume 261, July 2, 1993, © 1993 by the AAAS. Reprinted by permission.

Birkhäuser Boston: Excerpts from *The Mathematical Experience* by Philip J. Davis and Reuben Hersh © 1981 Birkhäuser Boston. Reprinted by permission of Birkhäuser Boston and the authors.

The Chronicle of Higher Education: Excerpts from *The Chronicle of Higher Education,* July 7, 1993, © 1993 *Chronicle of Higher Education.* Reprinted by permission.

The New York Times: Excerpts from *The New York Times,* June 24, 1993, © 1993 *The New York Times.* Reprinted by permission. Excerpts from *The New York Times,* June 29, 1993, © 1993 *The New York Times.* Reprinted by permission.

Cody Pfanstiehl: The poem on the subject of Fermat's last theorem is reprinted courtesy of Cody Pfanstiehl.

Karl Rubin, Ph.D.: The sketch of Dr. Wiles's proof of Fermat's Last Theorem included in the Appendix is reprinted courtesy of Karl Rubin, Ph.D.

Wesley Salmon, Ph.D.: Excerpts from *Zeno's Paradoxes* by Wesley Salmon, editor © 1970. Reprinted by permission of the editor.

Scientific American: Excerpts from "Turing Machines," by John E. Hopcroft, *Scientific American,* May 1984, © 1984 Scientific American, Inc. Reprinted by permission. Excerpts from "Sophie Germain," by Amy Dahan Dalmedico, *Scientific American,* December, 1991 © 1991, Scientific American, Inc. Reprinted by permission. Excerpts from "Fermat's Last Theorem," by Harold M. Edwards, *Scientific American,* October, 1978, © 1978 Scientific American, Inc. Reprinted by permission.

Springer-Verlag: Excerpts from *Thirteen Lectures on Fermat's Last Theorem* by Paulo Ribenboim © 1979 Springer-Verlag. Reprinted by permission of the publisher.

Time, Inc.: Excerpts from *Time* Magazine, July 5, 1993, © 1993 Time, Inc. Reprinted by permission.

Design by Sara Stemen

ISBN 0-312-10657-2

10 9 8 7 6 5 4 3 2

The World's Most Famous Math Problem

Dedicated to
Isaac Asimov,
with thanks for the memories.

Acknowledgments

I WANT TO thank the incomparable Martin Gardner for reading my manuscript, for being my dear long-distance friend, and for brightening the world for all thinking people. Author of the Mathematical Games column for *Scientific American* from 1956 to 1981, he wrote the brilliant *The Annotated Alice*—in which he showed how Lewis Carroll's novels actually were structured with chess games, coded messages, and diaphanous caricatures—as well as dozens of other books. I also want to thank Robert Weil, my editor at St. Martin's Press, for having the courage to ask that I write this book in three weeks. If it weren't for him, I wouldn't have known I could do it.

And a personal "thank you" to Barry Mazur, Kenneth Ribet, and Karl Rubin for being such good sports and for putting up with my faxes. Barry Mazur of Harvard University received his Ph.D. from Princeton University and is a member of the U.S. National Academy of Sciences; he has received the Vebien Prize

in Geometry and the Cole Prize in Number Theory from the American Mathematical Society. Mazur generously provided me with a text that evolved from his talk at the Symposium on Number Theory, held in Washington, D.C., under the auspices of the Board on Mathematical Sciences of the National Research Council.

Foreword

THIS ISN'T A math book. Instead, it's a book *about* math—about the possible end of a 350-year-long search and the start of a new one. Written for the non-mathematician, this book attempts to show the "queen of sciences" in a new way—as a science and as an art. You'll probably learn things about mathematics that you didn't know before—about its triumphs and failings, about its human aspects, and about its limits. Regardless of whether the new proof of Fermat's last theorem holds up under scrutiny, you'll learn how mathematics has thrown off the yoke of Euclid's legacy and ventured into the deepest waters of the imagination, whether for better or for worse. And you'll find it understandable to read, regardless of the extent of your mathematical education.

Karl Rubin of Ohio State University, who received his Ph.D. from Harvard University under the supervision of Andrew Wiles, and who was in attendance at the meeting where Wiles presented his proof of Fermat's last theorem, has generously allowed us to reprint his brief sketch of the highlights. (Rubin is best known for his work on elliptic curves, a special class of equations that play a fundamental role in the proof; he received the Cole Prize in Number Theory in 1992.) The sketch was sent through electronic mail to his math newsgroup following the last lecture, and we've left it intact in the appendix with all of its "you are there" charm, including the e-mail salutation and the translation idiosyncrasies, such as ^ instead of superscript. You'll find a not-very-plain-English version of the proof in the text.

The World's Most Famous Math Problem

Contents

ACKNOWLEDGMENTS ix
FOREWORD xi
[CHAPTER 1] ONE OF THE HOTTEST STORIES IN THE HISTORY OF MATH
Archimedes Thinking in the Bathtub 1
Pierre de Fermat Writing in the Margin 2
Andrew Wiles Lecturing at Cambridge University 3
The Five-Point Test 5
Dr. Wiles's Experience 8
Another Point to Ponder 11
Euclidean Versus the New Non-Euclidean Geometries 13
Doubling Cubes, Trisecting Angles, and Squaring Circles 17
[CHAPTER 2] PIERRE DE FERMAT AND THE LAST THEOREM
Pierre de Fermat 20
Fermat's Last Theorem 21
Learned Attempts at Proof 22
Unlearned Attempts at Proof 25
Who Is a Crank? 26
Wiles's Proof in Not-Very-Plain English 30
Unanswered Questions 31

[CHAPTER 3] PROOFS AND PUZZLES TO PONDER

What's a Theorem? 32
What's a Proof? 34
Faulty "Proofs" 36
Impractical Proofs 38
Branching Proofs 39
Philosophical Proofs 42
What Is Mathematical Truth? 44

[CHAPTER 4] MATHEMATICAL MYSTERIES, SOLVED AND UNSOLVED

The Parallel Postulate 46
Aristotle's Wheel 48
Zeno's Achilles 50
Zeno's Arrow 52
Zeno's Stadium 53
Kepler's Spheres 55
Goldbach's Conjecture 55
The Twin Prime Conjecture 55
Another Prime Conjecture 56

[CHAPTER 5] WHAT COMES NEXT

Why It All Matters 57
Why Wiles's Proof Isn't as Satisfying as It Could Be 60
A Possible Fatal Flaw 62
A Matter of Credit 62
Why This Proof Wasn't Fermat's Proof 63
To All the Unknown Genius in the World 64
The Search for Other Proofs 67
Fermat's Other Unproved Theorem 68

APPENDIX

Karl Rubin's Highlights of Andrew Wiles's Proof 73

FOR FURTHER READING 77

ABOUT THE AUTHOR 79

[1]

One of the Hottest Stories in the History of Math

[*Archimedes Thinking in the Bathtub*]

"EUREKA [I HAVE FOUND IT!] Then again, maybe I haven't." It's no wonder that Archimedes didn't say that second sentence while sinking down into a warm bath. In the legendary story from the third century B.C., the Greek tyrant Hiero II asked the renowed mathematician and physicist to find a method for determining whether a crown was made of pure gold or alloyed with silver instead. Considering Hiero's notoriously unpleasant temperament, Archimedes was lucky to realize, as he stepped into the tub, that a given weight of gold would displace less water than an equal weight of silver. (Gold is denser than silver, so a gold coin would be smaller than a silver coin that weighed the same.) In the throes of mathematical ecstasy over this momentous discovery, he is supposed to have run home stark naked, shouting "Eureka! Eureka!" ["I have found it! I have found it!"].

Archimedes was also lucky to be given a task that would yield to instant insight. (It's a good thing that showers—that "colonial abomination"—weren't popular then; mathematical progress might have been considerably slowed.) Proving Fermat's last theorem undoubtedly would have taken him considerably longer. But as of June of 1933, that puzzle has at last been solved. Well, maybe.

[*Pierre de Fermat Writing in the Margin*]

More than 350 years ago, a French mathematician and physicist named Pierre de Fermat wrote down an apparently simple little theorem in the margins of a mathematical book he was reading. The theorem was about what solutions are not possible for certain elementary equations. Fermat added that he had discovered a remarkable proof for his statement, but that there was no room in the margin to include it. He died without ever presenting the proof to substantiate this tantalizing claim. The best of mathematicians have been trying to do so ever since.

As Nigel Hawkes, Science Editor for *The Times of London*, put it on June 24, 1993, "Since Fermat wrote down the theorem, it has become the mathematical equivalent of Schubert's *Unfinished Symphony*." Two days later, the same newspaper labeled a proof of the theorem "the mathematical result of the century," calling it "as spectacular in its field as the discovery of Shakespeare's alleged *Love's Labour's Found* or the authentication of a lost Botticelli."

It's not as if no progress has been made at all. Since the arrival of computers, the theorem clearly has been shown to hold true for extremely high numbers. In 1992, an enormous computer effort verified Fermat's last theorem for exponents up to four million.

That might seem proof enough for the general public, but for mathematicians, it's no proof at all. Many of them were coming to the reluctant conclusion that Fermat's reach may have exceeded his grasp at last. "It has always been my belief that Fermat made a mistake," said Dr. Harold Edwards, a professor of mathematics at the Courant Institute of Mathematical Sciences at New York University and author of *Fermat's Last Theorem*, the definitive book on the subject, as quoted in *The Chronicle of Higher Education*. Still, mathematicians kept searching for the elusive proof.

Then, on July 2, 1993, came the news that sent shock waves through the mathematical community around the world. After dozens of claims of success made too early, hundreds made too unwisely, and thousands made too amateurishly, many mathematicians are cautiously heralding the work of a publically obscure, though highly respected colleague. They believe he may have at long last conquered what is perhaps the most intimidating test of strength and endurance and probably the most famous of unsolved problems in their corner of the intellectual landscape. (Mathematical details of Fermat's last theorem and an overview of Andrew Wiles's proposed proof can be found in Chapter Two. An abbreviated sketch of the proof itself can be found in the Appendix to this book.)

[*Andrew Wiles Lecturing at Cambridge University*]

Dr. Andrew Wiles is a reserved, bespectacled 40-year-old English mathematician at Princeton University. (It goes without saying that he's also very intelligent; there are no mathematicians who *aren't*.) Wiles became the subject of intense interest when he made a modest announcement to about seventy-five mathematicians at the end of a three-day lecture series at the newly opened Isaac Newton Institute for Mathematical Sciences at Cambridge University, in England, where he had done his doctoral studies.

This is how *Science* magazine decribed the atmosphere as the lectures were about to begin: "He was equally quiet when he arrived at the Newton Institute . . . but rumors of a breakthrough were starting to fly among the other participants—in part because Wiles, who normally doesn't ask to give lectures, had asked to give not just one, but three hour-long talks. John Coates of Cambridge University, who was Wiles's thesis adviser at Cambridge in the mid-1970s, scheduled him on Monday, Tuesday, and Wednesday, June 21–23."

Time magazine reported on the growing excitement in the hall: "By the end of the first hour . . . , they knew something was up. Recalls Nigel Boston, a visiting mathematician [there]: 'We realized where he could be heading. People were giving each other wide-eyed looks.' By the end of the third hour, the room was packed with excited number theorists. Wiles finished up his talk and wrote a simple equation on the blackboard, a mathematical afterthought that logically followed from all that he had been saying."

According to *The Associated Press* report on the following day, "Dr. Peter Goddard, the institute's deputy director, said . . . that nobody knew [Wiles] had come up with a solution until the very end of a lecture in which he had written down many mathematical results. 'He wrote down the last line, and the last line was a corollary to his last result, and it was Fermat's last theorem,' Goddard said. 'Then, he turned to the audience and smiled and said something like, 'I better stop there.' "

Reuter's reported, "There was a moment of stunned silence followed by rapturous applause as the enormity [sic] of the event sank in." Telephones began to ring, faxes churned out copy, electronic mail zapped into computers all over the world, and the communications satellites went into overdrive.

Could it really be true? Mathematicians were first startled, then excited, and ultimately—well, the emotions are mixed. One day after the event, *The New York Times* quoted Dr. Leonard Adelman of the University of Southern California as saying, "It's the most exciting thing that's happened in—geez—

maybe ever, in mathematics." But on June 29, 1993, the paper ran a more reflective response from Wiles himself.

> There is a certain sadness in solving the last theorem. All number theorists, deep down, feel that. For many of us, his problem drew us in, and we always considered it something you dream about, but never actually do. There is a sense of loss, actually.

[*The Five-Point Test*]

Surely, a great many mathematicians must share Wiles's sense of loss. But, just as surely, many may lack his concurrent elation. For them—and also for Wiles, of course—there are several further observations that merit consideration. The following is my own general guide to the analysis and ultimate evaluation of a proof. I call it the "Five-Point Test."

[1] *Might there be a subtle error in the proof?*

Yes. The two-hundred-page proof is unpublished as yet. Until it appears in mathematical journals, which could take months, and until it is thoroughly checked and rechecked and double-checked, there remains the possibility that it contains a significant error. (There are also likely to be a few minor errors that can be corrected without damaging the overall proof. This is called "polishing" the proof.)

[2] *Does the proof rest on numerous other proofs (especially recent, arcane, obscure, or delicate ones)?*

Yes. In 1954, the late Japanese mathematician Dr. Yutaka Taniyama made a conjecture about elliptic curves (a particular class of cubic equations). Taniyama's conjecture (known also as Taniyama-Weil for broader attribution) was further clarified by Dr. Goro Shimura of Princeton University. (The conjecture now can be known as Shimura-Taniyama-Weil, but for simplicity's sake, further references