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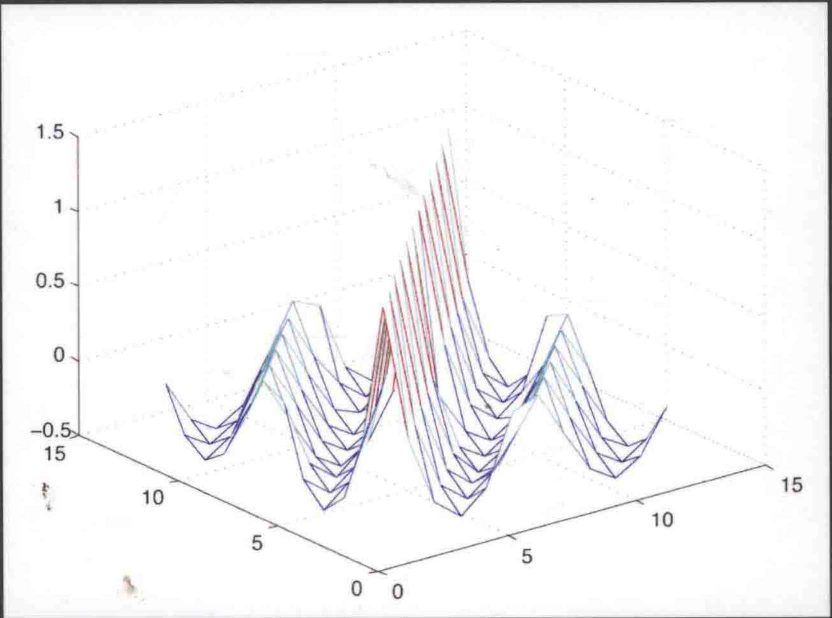
GROUP INVERSES OF M-MATRICES AND THEIR APPLICATIONS



Stephen J. Kirkland
Michael Neumann

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To Seema

–S.J.K.

To Helen

–M.N.

Preface

Nonnegative matrices and M-matrices have become a staple in contemporary linear algebra, and they arise frequently in its applications. Such matrices are encountered not only in matrix analysis, but also in stochastic processes, graph theory, electrical networks, and demographic models. We, the authors have worked for many years investigating these topics, and have found group inverses for singular M-matrices to be a recurring and useful tool.

The aim of this monograph is to bring together a diverse collection of results on the group inverses of M-matrices, highlighting their importance and utility in this body of research. By presenting these results in a single volume, we hope to emphasise the connections between problems arising in Markov chains, Perron eigenvalue analysis, spectral graph theory, and the like, and to show how insight into each of these areas can be gained through the use of group inverses.

While we have tried to make this book as self-contained as possible, we assume that the reader is familiar with the basics on the theory of nonnegative matrices, directed and undirected graphs, and Markov chains. There are many excellent texts covering these subjects, and we will refer the reader to these as needed.

The structure of the book is as follows. As a way of motivating the questions considered in the remainder of the book, in Chapter 1 we pose some sample problems associated with Leslie matrices (which arise in a demographic model) and stochastic matrices (which are central in the theory of Markov chains). In Chapter 2, we develop the basic algebraic and spectral properties of the group inverse of a general matrix, with a specific focus on the case that the matrix in question is a singular and irreducible M-matrix. In Chapter 3, we consider the Perron value and vector of a nonnegative matrix as a function of the entries in that matrix, and derive formulas for their derivatives in terms of the group inverse of an associated M-matrix. These formulas are then applied in Chapter 4 to several classes of matrices, including the Leslie matrices mentioned above. The elasticity of the Perron value, which measures the contribution of an individual entry to the Perron value, is also considered, and the group inverse approach is used to analyse the derivatives of the elasticity with respect to the entries in the matrix.

In Chapters 5 and 6, we turn our attention to Markov chains. In Chapter 5, the group inverse of an appropriate M-matrix is used not only in the perturbation analysis of the stationary distribution vector, but also in deriving bounds on the asymptotic rate of convergence of the underlying Markov chain. In Chapter 6, we will show how the group inverse can be used to compute and analyse the mean first passage matrix for a Markov chain. An analogous approach is also used to discuss the Kemeny constant for a Markov chain.

Chapter 7 has a combinatorial flavour, and is devoted to the Laplacian matrix for an undirected graph. Here we use the group inverse of the Laplacian matrix in order to discuss the Weiner and Kirchhoff indices, as well as the algebraic connectivity of the graph. Applications to electrical networks are also considered.

Chapter 8 concludes the book with a discussion on computing the group inverse, and on the stability of doing so. Several approaches are considered and compared.

In writing this book, we were informed and inspired by the work of many. In particular, we benefited from the books of Adi Ben–Israel and Thomas Greville, Abraham Berman and Robert Plemmons, Stephen Campbell and Carl Meyer, and Hal Caswell. We should also single out the 1975 paper of Meyer, which serves as the starting point for many of the developments reported in this book. Over the years, we have had the pleasure of working with many talented co-authors, and a number of collaborative results appear in this book. As a show of our gratitude, let us list those collaborators here: Mahmud Akelbek, Minerva Catral, Yonghong Chen, Emeric Deutsch, Ludwig Elsner, Ilse Ipsen, Charles Johnson, Srinivasa Mohan, Jason Molitierno, Nic Ormes, K.G. Ramamurthy, Bryan Shader, Nun–Sing Sze, and Jianhong Xu.

S.J.K. and M.N., March, 2011.

The text above was written while Michael Neumann (Miki, to his friends) was visiting me in March of 2011. He and I had been working on the monograph long-distance for about seven months, and during his visit with me, we solidified the contents of the book, developed preliminary drafts of the first few chapters, and mapped out a plan for writing the remaining chapters. In April of 2011, Miki died, of natural causes, but very suddenly. The news of his passing came as a shock to me. We had been collaborators for some twenty years—more than that, we were good friends.

I resolved to complete the monograph on my own, attempting to stay true to both the content and the spirit of the project that Miki and I had envisioned together. What follows is the result, though it must be said that that the book would have been stronger had Miki been able to continue his work on it. The experience of completing this monograph on my own has been bittersweet for me: it has made me acutely aware of Miki's absence, but has also served as an extended farewell to my close friend.

While I was writing this book, my research was supported in part by the Science Foundation Ireland, under Grant No. SFI/07/SK/I1216b. Thanks are also due to Iarnród Éireann, as much of the book was written while I was availing myself of its service.

S.J.K., May, 2012.

Author Bios

Stephen Kirkland received a Ph.D. in Mathematics from the University of Toronto in 1989, having previously taken an M.Sc. in Mathematics from the University of Toronto (1985), and a B.Sc. (Honours) in Mathematics from the University of British Columbia (1984). He held postdoctoral positions at Queen's University at Kingston (1989–1991), and the University of Minnesota (1991–1992), and was a faculty member at the University of Regina from 1992 until 2009. He then moved to Ireland, and is currently a Stokes Professor at the National University of Ireland Maynooth. He is an editor-in-chief of the journal *Linear and Multilinear Algebra*, and serves on the editorial boards of several other journals. Kirkland's research interests are primarily in matrix theory and graph theory, with an emphasis on the interconnections between those two areas.

Michael Neumann received a B.Sc. in Mathematics and Statistics from Tel Aviv University in 1970, and a Ph.D. in Computer Science from London University in 1972. He subsequently held positions at the University of Reading (1972–1973), the Technion (1973–1975), the University of Nottingham (1975–1980) and the University of South Carolina (1980–1985). In 1985 he moved to the University of Connecticut, where he remained as a faculty member until his death in 2011. Over the course of his career, Neumann published more than 160 mathematical papers, primarily in matrix theory, numerical linear algebra and numerical analysis. He was elected as a member of the Connecticut Academy of Arts and Sciences in 2007, was named a Board of Trustees Distinguished Professor in 2007, and was appointed as the Stuart and Joan Sidney Professor of Mathematics in 2010.

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