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What is Mathematics

An Elementary Approach to Ideas and Methods

什么是数学

对思想和方法的基本研究

英文版·第2版

[美] Richard Courant Herbert Robbins 著

[英] Ian Stewart 修订

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What is Mathematics An Elementary Approach to Ideas and Methods

什么是数学 对思想和方法的基本研究（英文版 · 第2版）

"对整个数学领域中的基本概念和方法透彻清晰的阐述……通俗易懂。"

——爱因斯坦

本书是享有世界声誉的不朽名著，由Richard Courant和Herbert Robbins两位数学大家合著。原版初版于1941年，几十年来一直畅销不衰。书中充满了数学的奇珍异品，生动有趣地描绘出一幅数学世界的画卷，让你如入宝山，目不暇给。第2版由著名数学家Ian Stewart增写了新的一章，阐述了数学的最新进展，包括四色定理和费马大定理的证明等。

这是一本人人都能读的数学书，将为你开启一扇认识数学世界的窗口。无论你是初学者还是专家，学生还是教师，哲学家还是工程师，通过这本书，你都将领略到数学之美，最终迷上数学。

Richard Courant (1888－1972) 20世纪杰出的数学家，哥廷根学派重要成员。曾担任纽约大学数学系主任和数学科学研究院院长，为了纪念他，纽约大学数学科学研究院1964年改名为柯朗数学科学研究院，成为世界上最大的应用数学研究中心。他写的书《数学物理方程》为每一个物理学家所熟知，而他的《微积分学》也被认为是该学科的代表作。

Herbert Robbins（1915－2001）美国著名数学家和统计学家。他的研究涉及拓扑学、测度论、统计学等诸多领域。经验贝叶斯方法中的Robbins引理，图论中的Robbins定理，还有Robbins代数和Robbins问题都以他的名字命名。

Ian Stewart Warwick大学教授，著名数学家和科普作家，对灾变论作出了重要贡献，于2001年入选英国皇家学会。除本书外，他还撰写了*Concepts of Modern Mathematics*等许多名著

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内 容 提 要

本书是世界著名的数学科普读物. 它荟萃了许多数学的奇珍异宝，对数学世界做了生动而易懂的描述. 内容涵盖代数、几何、微积分、拓扑等领域，其中还穿插了许多相关的历史和哲学知识.

本书不仅是数学专业人员的必读之物，也是任何愿意做科学思考者的优秀读物. 对于中学数学教师、高中生和大学生来说，这都是一本极好的参考书.

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“毫无疑问，这本书将产生深远的影响，它应当人手一册，无论是专业人士还是愿意做科学思考的任何人.”

——《纽约时报》

“一部已臻极致的著作.”

——《数学评论》

“太妙了……这本书是巨大愉快和满足感的源泉.”

——《应用物理杂志》

“这本书是一部艺术著作.”

——Marston Morse（美国著名数学家）

“这是一部非常完美的著作……被数学家们视作生命线的一切基本思想和方法，这本书用最简单的例子就讲清楚了，实在令人惊讶.”

——Herman Weyl（著名数学家、物理学家）

20世纪的数学已经发展到让人望洋兴叹的地步，如何在一本可以带出去郊游时随便翻翻的作品中，把这门异常发达的学科的面貌体现在读者面前呢？柯朗的做法是搜集很多数学上的“珍品”，每个方面的讲述并非深不见底，但也不是蜻蜓点水. 适当地深入，然后在该结束的时候结束. 这种既非盲人摸象、亦非解剖大象的方法，可以让普通读者也能粗略领悟到数学无比精巧的结构之美……

好作品要让读者常读常新. 例如《西游记》，比起那些佛教典籍，太容易读懂了，但好玩的故事和浅显的文字背后，其思想上的玄妙实在不是一语、一人可以道破、穷尽的，故而历来评论绵绵不断；即便是普通读者，碰到一些社会现象，与小说中的情节做些类比，也有新的感悟. 那么科学著作能否也达到同样的功效呢？至少，《什么是数学》这本书是做到了.

——《中华读书报》

下面是一组来自互动网的书评

“刚读了几章，已经被震撼了……实在不知道怎么形容……对我来说，这本书就像是神作.”

“绝对的好书，应该每个人都有一本……”

“刚刚读了一章，感觉作者在很高的角度写《什么是数学》，既严谨又像科普读物，很喜欢里面的数学历史部分！”

“《什么是数学》是一部介绍数学科普知识的世界名著，不论是学生还是老师，都值得花一点时间来阅读，了解数学的基本概念和一些经典命题的历史及其背景.”

“这本书里所表达的思想的确比我们在学校学的数学思想要先进得多，什么是数学，这才是数学.”

“经典中的经典，非常棒的数学名著，我觉得本书每个人都应该读，特别是想理解世界是如何的人.数学就是应该如此真切地描述.”

版权声明

DEDICATED TO

ERNEST, GERTRUDE, HANS,

AND LEONORE COURANT

FOREWORD

In the summer of 1937, when I was a young college student, I was studying calculus by going through my father's book *Differential and Integral Calculus* with him. I believe that is when he first conceived of writing an elementary book on the ideas and methods of mathematics and of the possibility that I might help with such a project.

The book, *What is Mathematics?*, evolved in the following years. I recall participating in intensive editing sessions, assisting Herbert Robbins and my father, especially in the summers of 1940 and 1941.

When the book was published, a few copies had a special title page: *Mathematics for Lori*, for my youngest sister (then thirteen years old). A few years later, when I was about to be married, my father challenged my wife-to-be to read *What Is Mathematics*. She did not get far, but she was accepted into the family nonetheless.

For years the attic of the Courant house in New Rochelle was filled with the wire frames used in the soap film demonstrations described in Chapter VII, §11. These were a source of endless fascination for the grandchildren. Although my father never repeated these demonstrations for them, several of his grandchildren have since gone into mathematics and related pursuits.

No really new edition was ever prepared since the original publication. The revised editions referred to in the preface were essentially unchanged from the original except for a few corrections of minor errors and misprints; all subsequent printings have been identical to the third revised edition. In his last years, my father sometimes talked of the possibility of a major modernization, but he no longer had the energy for such a task.

Therefore I was delighted when Professor Ian Stewart proposed the present revision. He has added commentaries and extensions to several of the chapters in the light of recent progress. We learn that Fermat's Last Theorem and the four-color problem have been solved, and that infinitesimal and infinite quantities, formerly frowned upon as flawed concepts, have regained respectability in the context of "nonstandard analysis." (Once, during my undergraduate years, I used the word "infinity," and my mathematics professor said, "I won't have bad language in my class!") The bibliography has been extended to the present. We hope that this new edition of *What Is Mathematics?* will again stimulate interest among readers across a broad range of backgrounds.

Ernest D. Courant

Bayport, N. Y.
September 1995

PREFACE TO THE SECOND EDITION

What Is Mathematics? is one of the great classics, a sparkling collection of mathematical gems, one of whose aims was to counter the idea that "mathematics is nothing but a system of conclusions drawn from definitions and postulates that must be consistent but otherwise may be created by the free will of the mathematician." In short, it wanted to put the meaning back into mathematics. But it was meaning of a very different kind from physical reality, for the meaning of mathematical objects states "only the relationships between mathematically 'undefined objects' and the rules governing operations with them." It doesn't matter what mathematical things *are:* it's what they *do* that counts. Thus mathematics hovers uneasily between the real and the not-real; its meaning does not reside in formal abstractions, but neither is it tangible. This may cause problems for philosophers who like tidy categories, but it is the great strength of mathematics—what I have elsewhere called its "unreal reality." Mathematics links the abstract world of mental concepts to the real world of physical things without being located completely in either.

I first encountered *What Is Mathematics?* in 1963. I was about to take up a place at Cambridge University, and the book was recommended reading for prospective mathematics students. Even today, anyone who wants an advance look at university mathematics could profitably skim through its pages. However, you do not have to be a budding mathematician to get a great deal of pleasure and insight out of Courant and Robbins's masterpiece. You do need a modest attention span, an interest in mathematics for its own sake, and enough background not to feel out of your depth. High-school algebra, basic calculus, and trigonometric functions are enough, although a bit of Euclidean geometry helps.

One might expect a book whose most recent edition was prepared nearly fifty years ago to seem old-fashioned, its terminology dated, its viewpoint out of line with current fashions. In fact, *What Is Mathematics?* has worn amazingly well. Its emphasis on problem-solving is up to date, and its choice of material has lasted so well that not a single word or symbol had to be deleted from this new edition.

In case you imagine this is because nothing ever changes in mathematics, I direct your attention to the new chapter, "Recent Developments," which will show you just how rapid the changes have been. No, the book has worn well because although mathematics is still growing, it is the sort of subject in which old discoveries seldom become obsolete. You cannot "unprove" a theorem. True, you might occasionally find that a long-accepted proof is wrong—it has happened. But then it was never proved in the first place. However, new viewpoints can often render old proofs obsolete, or old facts no longer interesting. *What Is Mathematics?* has worn well because Richard Courant and Herbert Robbins displayed impeccable taste in their choice of material.

Formal mathematics is like spelling and grammar—a matter of the correct application of local rules. Meaningful mathematics is like journalism—it tells an interesting story. Unlike some journalism, the story has to be true. The best mathematics is like literature—it brings a story to life before your eyes and involves you in it, intellectually and emotionally. Mathematically speaking, *What Is Mathematics?* is a very literate work. The main purpose of the new chapter is to bring Courant and Robbins's stories up to date—for example, to describe proofs of the Four Color Theorem and Fermat's Last Theorem. These were major open problems when Courant and Robbins wrote their masterpiece, but they have since been solved. I do have *one* genuine mathematical quibble (see §9 of "Recent Developments"). I think that the particular issue involved is very much a case where the viewpoint has changed. Courant and Robbins's argument is correct, within their stated assumptions, but those assumptions no longer seem as reasonable as they did.

I have made no attempt to introduce new topics that have recently come to prominence, such as chaos, broken symmetry, or the many other intriguing mathematical inventions and discoveries of the late twentieth century. You can find those in many sources, in particular my book *From Here to Infinity,* which can be seen as a kind of companion-piece to this new edition of *What Is Mathematics?*. My rule has been to add only material that brings the original up to date—although I have bent it on a few occasions and have been tempted to break it on others.

What Is Mathematics?

Unique.

Ian Stewart

Coventry
June 1995

PREFACE TO THE REVISED EDITIONS

During the last years the force of events has led to an increased demand for mathematical information and training. Now more than ever there exists the danger of frustration and disillusionment unless students and teachers try to look beyond mathematical formalism and manipulation and to grasp the real essence of mathematics. This book was written for such students and teachers, and the response to the first edition encourages the authors in the hope that it will be helpful.

Criticism by many readers has led to numerous corrections and improvements. For generous help with the preparation of the third revised edition cordial thanks are due to Mrs. Natascha Artin.

R. Courant

New Rochelle, N. Y.
March 18, 1943
October 10, 1945
October 28, 1947

PREFACE TO THE FIRST EDITION

For more than two thousand years some familiarity with mathematics has been regarded as an indispensable part of the intellectual equipment of every cultured person. Today the traditional place of mathematics in education is in grave danger. Unfortunately, professional representatives of mathematics share in the responsibility. The teaching of mathematics has sometimes degenerated into empty drill in problem solving, which may develop formal ability but does not lead to real understanding or to greater intellectual independence. Mathematical research has shown a tendency toward overspecialization and overemphasis on abstraction. Applications and connections with other fields have been neglected. However, such conditions do not in the least justify a policy of retrenchment. On the contrary, the opposite reaction must and does arise from those who are aware of the value of intellectual discipline. Teachers, students, and the educated public demand constructive reform, not resignation along the line of least resistance. The goal is genuine comprehension of mathematics as an organic whole and as a basis for scientific thinking and acting.

Some splendid books on biography and history and some provocative popular writings have stimulated the latent general interest. But knowledge cannot be attained by indirect means alone. Understanding of mathematics cannot be transmitted by painless entertainment any more than education in music can be brought by the most brilliant journalism to those who never have listened intensively. Actual contact with the *content* of living mathematics is necessary. Nevertheless technicalities and detours should be avoided, and the presentation of mathematics should be just as free from emphasis on routine as from forbidding dogmatism which refuses to disclose motive or goal and which is an unfair obstacle to honest effort. It is possible to proceed on a straight road from the very elements to vantage points from which the substance and driving forces of modern mathematics can be surveyed.

The present book is an attempt in this direction. Inasmuch as it presupposes only knowledge that a good high school course could impart, it may be regarded as popular. But it is not a concession to the dangerous tendency toward dodging all exertion. It requires a certain degree of intellectual maturity and a willingness to do some thinking on one's own. The book is written for beginners and scholars, for students and teachers, for philosophers and engineers, for class rooms and libraries. Perhaps this is too ambitious an intention. Under the pressure of other work some compromise had to be made in publishing the book after many years of preparation, yet before it was really finished. Criticism and suggestions will be welcomed.

At any rate, it is hoped that the book may serve a useful purpose as a contribution to American higher education by one who is profoundly grateful for the opportunity offered him in this country. While responsibility for the plan and philosophy of this publication rests with the undersigned, any credit for merits it may have must be shared with Herbert Robbins. Ever since he became associated with the task, he has unselfishly made it his own cause, and his collaboration has played a decisive part in completing the work in its present form.

Grateful acknowledgement is due to the help of many friends. Discussions with Niels Bohr, Kurt Friedrichs, and Otto Neugebauer have influenced the philosophical and historical attitude; Edna Kramer has given much constructive criticism from the standpoint of the teacher; David Gilbarg prepared the first lecture notes from which the book originated; Ernest Courant, Norman Davids, Charles de Prima, Alfred Horn, Herbert Mintzer, Wolfgang Wasow, and others helped in the endless task of writing and rewriting the manuscript, and contributed much in improving details; Donald Flanders made many valuable suggestions and scrutinized the manuscript for the printer; John Knudsen, Hertha von Gumppenberg, Irving Ritter, and Otto Neugebauer prepared the drawings; H. Whitney contributed to the collection of exercises in the appendix. The General Education Board of the Rockefeller Foundation generously supported the development of courses and notes which then became the basis of the book. Thanks are also due to the Waverly Press, and in particular Mr. Grover C. Orth, for their extremely competent work; and to the Oxford University Press, in particular Mr. Philip Vaudrin and Mr. W. Oman, for their encouraging initiative and coöperation.

R. Courant

New Rochelle, N. Y.
August 22, 1941

HOW TO USE THE BOOK

The book is written in a systematic order, but it is by no means necessary for the reader to plow through it page by page and chapter by chapter. For example, the historical and philosophical introduction might best be postponed until the rest of the book has been read. The different chapters are largely independent of one another. Often the beginning of a section will be easy to understand. The path then leads gradually upward, becoming steeper toward the end of a chapter and in the supplements. Thus the reader who wants general information rather than specific knowledge may be content with a selection of material that can be made by avoiding the more detailed discussions.

The student with slight mathematical background will have to make a choice. Asterisks or small print indicate parts that may be omitted at a first reading without seriously impairing the understanding of subsequent parts. Moreover, no harm will be done if the study of the book is confined to those sections or chapters in which the reader is most interested. Most of the exercises are not of a routine nature; the more difficult ones are marked with an asterisk. The reader should not be alarmed if he cannot solve many of these.

High school teachers may find helpful material for clubs or selected groups of students in the chapters on geometrical constructions and on maxima and minima.

It is hoped that the book will serve both college students from freshman to graduate level and professional men who are genuinely interested in science. Moreover, it may serve as a basis for college courses of an unconventional type on the fundamental concepts of mathematics. Chapters III, IV, and V could be used for a course in geometry, while Chapters VI and VIII together form a self-contained presentation of the calculus with emphasis on understanding rather than routine. They could be used as an introductory text by a teacher who is willing to make active contributions in supplementing the material according to specific needs and especially in providing further numerical examples. Numerous exercises scattered throughout the text and an additional collection at the end should facilitate the use of the book in the class room.

It is even hoped that the scholar will find something of interest in details and in certain elementary discussions that contain the germ of a broader development.

WHAT IS MATHEMATICS?

Mathematics as an expression of the human mind reflects the active will, the contemplative reason, and the desire for aesthetic perfection. Its basic elements are logic and intuition, analysis and construction, generality and individuality. Though different traditions may emphasize different aspects, it is only the interplay of these antithetic forces and the struggle for their synthesis that constitute the life, usefulness, and supreme value of mathematical science.

Without doubt, all mathematical development has its psychological roots in more or less practical requirements. But once started under the pressure of necessary applications, it inevitably gains momentum in itself and transcends the confines of immediate utility. This trend from applied to theoretical science appears in ancient history as well as in many contributions to modern mathematics by engineers and physicists.

Recorded mathematics begins in the Orient, where, about 2000 B.C., the Babylonians collected a great wealth of material that we would classify today under elementary algebra. Yet as a science in the modern sense mathematics only emerges later, on Greek soil, in the fifth and fourth centuries B.C. The ever-increasing contact between the Orient and the Greeks, beginning at the time of the Persian empire and reaching a climax in the period following Alexander's expeditions, made the Greeks familiar with the achievements of Babylonian mathematics and astronomy. Mathematics was soon subjected to the philosophical discussion that flourished in the Greek city states. Thus Greek thinkers became conscious of the great difficulties inherent in the mathematical concepts of continuity, motion, and infinity, and in the problem of measuring arbitrary quantities by given units. In an admirable effort the challenge was met, and the result, Eudoxus' theory of the geometrical continuum, is an achievement that was only paralleled more than two thousand years later by the modern theory of irrational numbers. The deductive-postulational trend in mathematics originated at the time of Eudoxus and was crystallized in Euclid's *Elements*.

However, while the theoretical and postulational tendency of Greek mathematics remains one of its important characteristics and has exercised an enormous influence, it cannot be emphasized too strongly