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Classification and Identification of Lie Algebras

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American Mathematical Society

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Classification and Identification of Lie Algebras

Preface

The purpose of this book is to serve as a tool for practitioners of Lie algebra and Lie group theory, i.e., for those who apply Lie algebras and Lie groups to solve problems arising in science and engineering. It is not intended to be a textbook on Lie theory, nor is it oriented towards one specific application, for instance the analysis of symmetries of differential equations. We restrict our attention to finite-dimensional Lie algebras over the fields of complex and real numbers.

In any application Lie algebras typically arise as sets of linear operators that commute with a given operator, say the Hamiltonian of a physical system. Alternatively, Lie groups arise as groups of (local) transformations leaving some object invariant; the corresponding Lie algebra then consists of vector fields generating 1-parameter subgroups. The object may be for instance the set of all solutions of a system of equations. The equations can be differential, difference, algebraic or integral ones, or some combination of such equations. They may be linear or nonlinear. In any case, the Lie algebra is realized by some operators in a basis that is usually not the standard one and that depends crucially on the manner in which it was obtained. The structure constants of Lie algebras can be calculated in any basis, but they in turn are basis dependent and reveal very little about the actual structure of the given Lie algebra.

After the Lie algebra $\mathfrak{g}$ associated with a studied problem is found, the next task that faces the researcher is to identify the Lie algebra as an abstract Lie algebra. In some cases $\mathfrak{g}$ may be isomorphic to a known algebra given in some accessible list. This is certainly the case for semisimple Lie algebras in view of Cartan's classification of all simple Lie algebras over the complex numbers, and subsequent classification of their real forms.

The fundamental Levi theorem, stating that every finite dimensional Lie algebra is isomorphic to a semidirect sum of a semisimple Lie algebra and the maximal solvable ideal (the radical) greatly simplifies the task of identifying a given Lie algebra. The weak link is that no complete classification of solvable Lie algebras exists, nor can one be expected to be produced in the future.

The problem addressed in this book is that of transforming a randomly obtained basis of a Lie algebra into a "canonical basis" in which all basis independent features of the Lie algebra are directly visible. For low dimensional Lie algebras (of dimension less or equal six) this makes it possible to identify the Lie algebra completely. In this book we give a representative list of all such Lie algebras. As stated above, in any dimension a complete identification can be performed for semisimple Lie algebras. We also describe some classes of nilpotent and solvable Lie algebras of arbitrary finite dimensions for which a complete classification exists and hence an exact identification is possible.

The book has four parts. The first presents some general results and concepts that are used in the subsequent chapters. In particular such invariant notions as the dimension of ideals in the characteristic series, and the invariants of the coadjoint representation are introduced.

In Part 2 we present algorithms that accomplish the following tasks:

(1) An algorithm for determining whether the algebra $\mathfrak{g}$ can be decomposed into a direct sum. If $\mathfrak{g}$ is decomposable the algorithm provides a basis in which $\mathfrak{g}$ is explicitly decomposed into a direct sum of indecomposable Lie subalgebras.

(2) A further algorithm is presented to find the radical $R(\mathfrak{g})$ and the Levi factor, i.e., the semisimple component of $\mathfrak{g}$.

(3) If the Lie algebra is solvable, for instance if it is the radical of a larger algebra, then it is necessary to identify its nilradical, i.e., the maximal nilpotent ideal. A rational (i.e., avoiding calculation of eigenvalues) algorithm for performing this is presented.

The text includes many examples illustrating various situations that may arise in such computations. All these algorithms have been implemented on computers.

Part 3 is devoted to solvable and nilpotent Lie algebras. While a complete classification of such algebras seems not to be feasible, it is possible to take a class of nilpotent Lie algebras and construct all extensions of these algebras to solvable ones. Finite-dimensional solvable Lie algebras with Abelian, Heisenberg, Borel, filiform and quasifiliform nilradicals are presented in Part 3.

Part 4 of the book consists of tables of all indecomposable Lie algebras of dimension n where $1 \leq n \leq 6$. They are ordered in such a way as to make the identification of any given low-dimensional Lie algebra written in an arbitrary basis as simple as possible. Any Lie algebra up to dimension 6 is isomorphic to precisely one entry in the tables. Essential characteristics of each algebra including its Casimir invariants are also provided.

The book is based on material that was previously dispersed in journal articles, many of them written by one or both of the authors of this book together with collaborators. The tables in Part 4 are based on older results and have been independently verified, in some cases corrected, unified and ordered by structural properties of the algebras (rather than by the way they were originally obtained).

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L. Šnobl dedicates this book to his parents Libuše and Zdeněk. P. Winternitz dedicates this book to his wife Milada and his sons Peter and Michael. We both thank them for their support and encouragement.

Contents

Part 1

General Theory

CHAPTER 1

Introduction and Motivation

Lie groups and Lie algebras appear in science in many different guises. They may be a priori parts of the theory, like Lorentz or Galilei invariance of most physical theories, or the (semi)simple Lie groups of the Standard model in particle theory.

Alternatively, specific Lie groups may appear as consequences of specific dynamics. Consider any physical system with dynamics described by a system of ordinary or partial differential equations. This system of equations will be invariant under some local Lie group of local point transformations, taking solutions into solutions. This symmetry group G and its Lie algebra $\mathfrak{g}$ can be determined in an algorithmic manner [86]. The Lie algebra $\mathfrak{g}$ is obtained as an algebra of vector fields, usually in some nonstandard basis, depending on the way in which the algorithm is applied.

An immediate task is to identify the algebra found as being isomorphic to some known abstract Lie algebra. To do this we must transform it to a canonical basis in which all basis independent properties are manifest. Thus, if $\mathfrak{g}$ is decomposable into a direct sum, it should be explicitly decomposed into components that are further indecomposable

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k. \tag{1.1}$$

Each indecomposable component must be further identified. Let $\mathfrak{g}$ now denote such an indecomposable Lie algebra. A fundamental theorem due to E. E. Levi [59, 71] tells us that any finite-dimensional Lie algebra can be represented as the semidirect sum

$$\mathfrak{g} = \mathfrak{p} \niplus \mathfrak{r}, \quad [\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}, \quad [\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r}, \quad [\mathfrak{p}, \mathfrak{r}] \subseteq \mathfrak{r}, \tag{1.2}$$

where the subalgebra $\mathfrak{p}$ is semisimple and $\mathfrak{r}$ is the *radical* of $\mathfrak{g}$, i.e., its maximal solvable ideal. If $\mathfrak{g}$ is simple, we have $\mathfrak{r} = 0$ (an indecomposable semisimple algebra is actually simple). If $\mathfrak{g}$ is solvable, we have $\mathfrak{p} = 0$. Algorithms realizing decompositions (1.1), (1.2) exist [102] and are presented below.

In view of the Levi theorem (1.2) the classification of all finite-dimensional Lie algebras can be reduced to three steps:

(1) Classification of all simple Lie algebras. This also provides a classification of all semisimple ones.
(2) Classification of all solvable Lie algebras.
(3) Classification of all possible linear actions of the semisimple algebra $\mathfrak{p}$ on the radical $\mathfrak{r}$.

Semisimple Lie algebras over the field of complex numbers $\mathbb{C}$ have been completely classified by W. Killing and É. Cartan [22, 60], over the field of real numbers $\mathbb{R}$ by É. Cartan in [20, 21] (the analysis in the real case was later simplified by F. Gantmacher in [50]).

The third step is basically a task of the representation theory of semisimple Lie algebras. The main unsolved problem is the classification of all finite dimensional solvable Lie algebras, a task that does not have a realistic solution. Even all nilpotent Lie algebras are impossible to classify. (For instance in [127] the authors claim that they have found 24 168 nonisomorphic 9-dimensional nilpotent algebras Lie algebras with a maximal Abelian ideal of dimension 7 alone.) A realistic partial classification problem is to classify all solvable Lie algebras with a given nilradical of an arbitrary finite dimension n. So far this has been done for certain series of nilpotent Lie algebras, namely Abelian, Heisenberg and Borel nilradicals, as well as certain filiform and quasifiliform algebras. The results of the original articles [83, 84, 106, 115–118, 123, 124] are presented in a unified manner in Part 3 of this book.

An interesting physical application of the classification of low-dimensional Lie algebras is in general relativity. Indeed, the classification of Einstein spaces according to their isometry groups [95] is based on the work of Bianchi and his successors [6, 65]. The Petrov classification concerns Einstein spaces of dimension 4 and hence involves isometry groups of relatively low dimensions [95, 120]. String theory, brane cosmology and some other elementary particle theories going beyond the standard model require the use of higher-dimensional spaces. Any attempt at a Lie group classification of such spaces will require knowledge of higher-dimensional Lie groups, including solvable ones.

One of the very useful applications of Lie group analysis in science is the identification of seemingly different problems that are mathematically equivalent. Indeed, let us consider two systems, say (A) and (B), of differential equations

$$\begin{gathered} E_a^A(\vec{X},\vec{U};\partial_{X_i}U_\alpha,\partial^2_{X_iX_j}U_\alpha,\dots)=0,\\ E_a^B(\vec{x},\vec{u};\partial_{x_i}u_\alpha,\partial^2_{x_ix_j}u_\alpha,\dots)=0,\\ \vec{X}=(X_1,\dots,X_p),\qquad \vec{U}=(U_1,\dots,U_q),\\ \vec{x}=(x_1,\dots,x_p),\qquad \vec{u}=(u_1,\dots,u_q),\\ 1\le a\le N,\quad 1\le\alpha\le q,\quad 1\le i,j,\dots\le p \end{gathered}\tag{1.3}$$

of the same order describing different physical processes. From the mathematical viewpoint, we may consider these two systems equivalent if there exists a local invertible transformation of the independent and dependent variables

$$X_i=\Lambda_i(\vec{x},\vec{u}),\qquad U_\alpha=\Omega_\alpha(\vec{x},\vec{u})\tag{1.4}$$

transforming the systems (A) and (B) into each other. The transformation (1.4) together with the mapping between the spaces of functions $\vec{u}(\vec{x})$ and $\vec{U}(\vec{X})$ induced by it is called a *point transformation*.

A necessary condition for the equivalence of the systems (A) and (B) is that they have isomorphic Lie algebras of infinitesimal point symmetries $\mathfrak{g}_A$ and $\mathfrak{g}_B$. The algebras $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are by construction realized by vector fields in variables $(\vec{X},\vec{U})$ and $(\vec{x},\vec{u})$, respectively.

If $\mathfrak{g}_A$ and $\mathfrak{g}_B$ are isomorphic then a local point transformation (1.4) may exist such that it transforms the vector fields of $\mathfrak{g}_A$ into those of $\mathfrak{g}_B$ and vice versa. The transformation is unique up to point transformations leaving $\mathfrak{g}_A$ or $\mathfrak{g}_B$ invariant. In any case, the transformation taking $\mathfrak{g}_A$ into $\mathfrak{g}_B$ will also take the system (A)