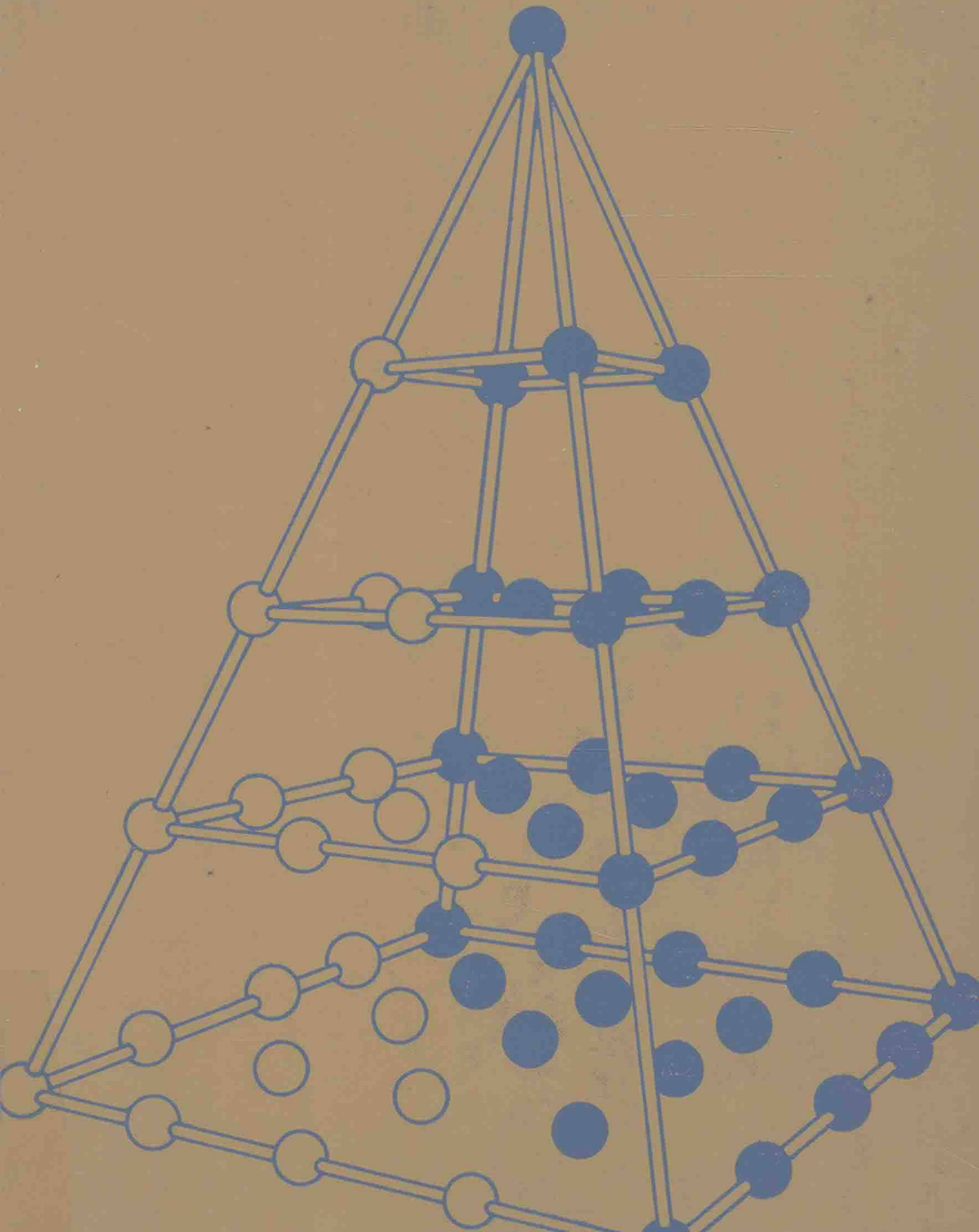
MATHEMATICS

AN EXPLORATORY APPROACH

ROBERT G. STEIN



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McGRAW-HILL BOOK COMPANY
New York St. Louis San Francisco Düsseldorf Johannesburg Kuala Lumpur
London Mexico Montreal New Delhi Panama Paris São Paulo Singapore
Sydney Tokyo Toronto

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1234567890 SGDO 798765

This book was set in Times Roman by Black Dot, Inc. The editors were Stephen D. Dragin and Barry Benjamin; the designer was Jo Jones; the production supervisor was Dennis J. Conroy. The drawings were done by ANCO Technical Services.
The printer was Segerdahl Corporation; the binder, R. R. Donnelley & Sons Company.

Library of Congress Cataloging in Publication Data

Stein, Robert G.
Mathematics: an exploratory approach.

Bibliography: p.
1. Mathematics—1961- I. Title.
QA39.2.S72 510 74-20635
ISBN 0-07-060993-4

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To my wife, Ronnie

PREFACE

This book is neither an old-fashioned "cookbook" treatment nor its usual replacement, a "modern" course in mathematics emphasizing logical structure and deduction. Instead this book is intuitive and inductive, viewing mathematics primarily as a kind of activity rather than a body of knowledge. In spirit it is somewhere between the works of George Polya and W. W. Sawyer. Wherever possible it asks rather than tells, seeking to involve the student in raising mathematical questions and attempting to answer them. To this end many ideas are presented through experiments, both in the exercises and in the text itself. Naturally, this makes the book especially suitable for activity-oriented courses.

A particularly important consideration underlying this book is that many prospective elementary teachers dislike mathematics, are weak at it, and often try to avoid it. Therefore every effort has been made to present the material in a way that will involve the student without threatening him. In this the first lesson is crucial; it sets the tone for the entire book and strongly influences the frame of mind the student will bring to subsequent material. Thus instead of beginning with sets and numeration, as the logic of the material might suggest, this book starts with an open-ended exploration designed to involve the student and show him that he can indeed understand mathematics.

To maintain this attitude in later sections, even in more sophisticated material, jargon, symbolism, and abstraction are minimized, and generality and rigor are sacrificed for concreteness and clarity. Explanations rely freely on diagrams and are phrased in terms of particular cases instead of general cases. To reinforce the view of mathematics as a living subject, problems are mentioned whose solutions are not known today, and many historical references are included.

The approach to mathematics used in this book is consistent with recent work in the psychology of learning, which stresses the importance of introducing mathematical ideas concretely and progressing only gradually to abstractions. This means that the material is presented here in a way that is particularly useful for prospective teachers, because it can be readily adapted for use in elementary schools.

There is ample material here for a semester, but it can easily be adapted for courses lasting only a quarter. Perhaps the best way to do this is to treat the starred sections and problems as optional enrichment material, though students who will teach only the primary grades may skip Chaps. 12, 14, and 15 entirely, allowing a more leisurely treatment of the rest.

I owe a great debt of thanks to all those who helped me write this book. This includes many colleagues and students, especially Rochelle Campbell, who encouraged me, helped prepare preliminary versions for class testing, and suggested improvements. I must also thank the outstanding teachers I have known, especially Hans Hollstein and W. W. Sawyer, for their inspiration. Last and by far most, I thank my wife, Ronnie, who bore through the long project patiently and used her knowledge of teaching to suggest many improvements.

Robert G. Stein

TO THE READER

A famous college football coach once remarked, "Ninety percent of the job is getting the kids on your side." This applies to all teaching, but in mathematics it is easier said than done, because the subject is widely disliked. This book is designed to change that; one of its main goals is to show how elementary mathematics can be presented in an interesting way. This book is designed to build up your mathematical background; and if you teach, you can use it as a source of specific ideas for your own classroom.

Here you will find nothing to memorize and no repetitious drill. Instead you will find yourself challenged to explore mathematical questions, using your imagination and creativity. You may be surprised to find how much of mathematics you can discover for yourself. Of course exploration cannot be done passively: you are involved actively throughout. Read with pencil in hand, guessing at patterns and testing ideas as you go.

The exercises are a particularly important part of the book. Some are routine, designed to help you practice and test yourself on material discussed in the text. Others lead you to discoveries, amplifying the text or taking up points not mentioned there. Do no more of the routine exercises than you need for your own self-confidence. Concentrate instead on the more interesting exploration and pattern exercises. Do as many of these as you can, using any shortcuts and clever ideas that occur to you. (Ample practice in basic skills is built into these exploration exercises.) A star by a problem indicates that it is either especially challenging or a bit off the beaten path; skip it if it does not interest you. Similarly, starred portions of the text are included primarily as enrichment. They may be skipped with no loss of continuity, but you will probably find among them some of the most interesting and enjoyable parts of the book.

As you read and as you try to use ideas from this book in your own classroom, you will doubtless find places where it can be improved. Your suggestions and comments will be most welcome.

Robert G. Stein

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*See preface for explanation on starred material.

CHAPTER 1
ADDING AND MULTIPLYING

Mathematics is an activity of exploration and investigation. This chapter will involve you in such activity. Do not be afraid if your arithmetic is rusty; the work in this chapter involves simple numbers and has built-in checks, and in the course of the book arithmetic is reviewed in some detail.

1.1 BOX PUZZLE EXPERIMENTS

Adding across the rows and down the columns of a little box like Fig. 1.1 leads to some interesting discoveries. Adding across the top row, we put 10, which is 2 + 8, in the upper right-hand corner and 12, which is 7 + 5, in the square just below the 10, as in Fig. 1.2. Next we add our original numbers down the columns to get Fig. 1.3, leaving only the lower right-hand corner empty. How should it be filled in?

2	8	
7	5	

Figure 1.1

2	8	10
7	5	12

Figure 1.2

2	8	10
7	5	12
9	13	

Figure 1.3

2	8	10
7	5	12
9	13	22

Figure 1.4

Adding down the right-hand column yields $10 + 12 = 22$, and adding across the bottom row yields $9 + 13$, which is also 22. Since both ways yield the same result, we complete the box as in Fig. 1.4.

That the box is now filled in may signify the end of the exercise, but the inquisitive always look for new questions to ask. Questions are the key to discovery. When adding down the right-hand column leads to the same number,

3	13	
10	12	

Figure 1.5

3	13	16
10	12	22
13	25	

Figure 1.6

22, as adding across the bottom row, is this a numerical coincidence or something more fundamental? Since a good starting point for any investigation is to gather more evidence, we repeat the experiment with different numbers. Starting with Fig. 1.5 leads to Fig. 1.6. Now adding down the right-hand column yields 38, and so does adding across the bottom row.

There does indeed seem to be something to investigate. Is there any significance to the fact that both examples began with two even numbers and two odd numbers? What if one or more of the numbers were zeros? What if larger numbers or fractions are involved? (If you do not know how to add fractions, omit this part.) What if the format is changed, for example to something like the puzzles shown in Fig. 1.7? What if addition is replaced throughout by multiplication? Before reading on, try to answer these questions for yourself by making up examples and observing the results. Try to find out when the "coincidence" happens and when it does not, and why. When you have experimented and pondered a bit, read on.

3	11	2	
15	4	7	
8	6	10	

5	10	13	
1	17	20	

Figure 1.7

Are you convinced that more than coincidence is involved? Did you notice that when each box is complete, the number in the lower right-hand corner is the sum of the numbers with which the box was begun? (The exception occurs when addition is replaced by multiplication, and then the number is the result of multiplying the original numbers, which is called their *product.*) In our first example the number 22 is the sum of the original four numbers 2, 8, 7, and 5, though this sum was found two ways. Adding first across the rows and then down the right-hand column yields $(2 + 8) + (7 + 5)$. The parentheses indicate grouping; whatever is inside a pair of parentheses is to be dealt with as a single number, so that $(2 + 8) + (7 + 5)$ is really $10 + 12$. Adding first down

a	b	
c	d	

Figure 1.8

a	b	$a+b$
c	d	$c+d$

Figure 1.9

a	b	
c	d	
$a+c$	$b+d$	

Figure 1.10

the columns and then across the bottom row yields $(2+7)+(8+5)$ or $9+13$.

Similar reasoning applies to other examples. If the four numbers are called a, b, c, and d, as in Fig. 1.8, adding across the rows yields Fig. 1.9, and adding down the columns yields Fig. 1.10. Adding down the right-hand column of Fig. 1.9 yields $(a+b)+(c+d)$, and adding across the bottom row of Fig. 1.10 yields $(a+c)+(b+d)$. Can you complete the reasoning from here? The key to the investigation is that $(a+b)+(c+d)$ is the same as $(a+c)+(b+d)$ no matter what numbers a, b, c, and d stand for. This is true because if only addition is involved, neither grouping nor order has any bearing on the result. The "coincidence" observed earlier will occur regardless of what numbers you start with because it depends on two basic properties of addition.

The fact that grouping does not affect a sum is known as the *associative* property of addition, since it deals with how numbers to be added are grouped, or associated. Symbolically, for any three numbers p, q, and r,

$$(p+q)+r=p+(q+r)$$

Because addition is associative, sums of more than two numbers may be written without grouping symbols since no matter how they are grouped the sum will be the same.

In our statement of the associative property, the numbers p, q, and r are written in the same order (left to right; first p, then q, then r) both times. That a sum is independent of the order in which the numbers are added is summarized in a second basic property, the *commutative* property of addition. Symbolically, if s and t are any numbers whatever, then

$$s+t=t+s$$

Addition is not the only process which is both associative and commutative. Multiplication is too. Therefore it is not surprising that the box puzzles still work when addition is replaced throughout by multiplication. The associative and commutative properties, like most "new mathematics," are old. Chrystal's "Textbook of Algebra," first published in 1882, was one of the first books to treat numbers in terms of these and a few other basic laws. It is a reasonable teaching approach, as it simplifies the subject from a collection of seemingly unrelated facts and operations to be memorized to a few general principles.

Recently this approach has been tried in elementary schools, but there have been problems in carrying it out. One is the confusion of words with the ideas for which they are supposed to stand. Most elementary school children find words like "commutative" and "associative" hard; they do not need such words. They do need to know, operationally, the concepts behind the words. That is, they need to know that order and grouping do not affect sums and products, and they need to know how to use these ideas in computation. Words should be introduced only where they help, and they should be kept simple and clear. That is the general approach in this book, though some words are introduced not because they are necessary in themselves but because they are widely used in today's elementary school books.

A direct, intuitive explanation of box puzzles for addition is available, as long as fractions, negative numbers, or other complications are avoided, for then addition can be interpreted in terms of merging collections of objects. (This operation, known technically as the *union of sets*, is discussed more fully in the Appendix.) Think of the numbers as representing numbers of objects in each square, so that Fig. 1.11 represents the puzzle in Fig. 1.1. With this interpretation, adding across the top row may be thought of as putting all the objects from the top row (the two circles and the eight crosses) into the upper right-hand corner. Similarly adding across the middle row will put the seven squares and five triangles into the right column, as in Fig. 1.12. Then adding down the right-hand column puts all the original objects into the lower right-hand corner, explaining why the number which appears there is the sum of the four numbers we started with. If instead we had begun by adding down the columns and then across the bottom row, we still would have ended up with all the original objects in the lower right-hand corner, showing why this number can be obtained either by adding across the rows first or down the columns first. This reasoning holds regardless of how many objects appear in each square. It even holds for box puzzles like those in Fig. 1.7, in which more squares are involved.

Box puzzles are an excellent way to teach the associative and commutative laws intuitively. At the same time they offer practice in computation, not as drill but as part of a larger investigation. Students need computation practice and enjoy it when it is not boring; in this case the children make up puzzles and experiments suitable to their own level of difficulty. The teacher should not spoil the fun by leading them to an explanation too soon, because it robs them of practice and prevents them from reaching their own understanding. Besides,

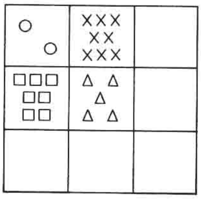

Figure 1.11

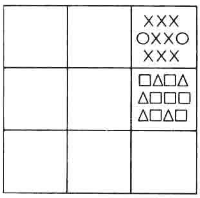

Figure 1.12

it takes students time to recognize that there is something worth investigating. Explaining too soon is like writing a whodunit and showing in chapter 1 that the butler did it. After that there is no point to the rest. We shall see some other "mysteries" about box puzzles, but these involve the basic relationship between multiplication and addition, which we pause to consider first.

Exercises 1.1

1 The commutative and associative properties of addition are often used (usually by people who do not realize it) to simplify mental arithmetic. For example, to add $87 + 71 + 13$, one might first add 87 to 71 to get 158, then add 13 to that to get 171. This amounts to grouping $87 + 71 + 13$ as $(87 + 71) + 13$. But if you notice that $87 + 13 = 100$, you can see right away that the sum is 171. The procedure can be justified by these successive applications of the associative and commutative properties:

Step 1: $(87 + 71) + 13 = 87 + (71 + 13)$

Step 2: $= 87 + (13 + 71)$

Step 3: $= (87 + 13) + 71$

Which of these steps involve the associative property and which the commutative property?

2 Does the English language have the associative property? Ella Wheeler Wilcox, a journalist who took herself seriously as a poet, was furious when her line

My soul is a lighthouse keeper

appeared in print as

My soul is a light housekeeper

A more mundane example can be worked with "Is your doghouse broken?" Can you think of others?

1.2 ON MULTIPLICATION

In the simplest cases, which do not involve fractions or other complications, multiplication may be regarded as an abbreviation for repeated addition of the same number. For example, $3 + 3 + 3 + 3 + 3$ is 3 added 5 times, which is called "5 times 3" or "five 3s." Finding this sum is called "multiplying 5 by 3." A number which results from multiplication is called a *product*, and the numbers that are multiplied are called *factors*. In our example, addition shows that the product of 5 and 3 is 15.

In elementary schools the symbol × is usually used for multiplication, but in more advanced work a centered dot is often used, perhaps to avoid confusing × with x. We shall usually use the dot. Note that it is written high enough not to look like decimal point.

Counting the objects in a rectangular array is sometimes a helpful way to