STUDIES IN LOGIC
AND
THE FOUNDATIONS OF MATHEMATICS

VOLUME 100

Descriptive Set Theory

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NORTH-HOLLAND PUBLISHING COMPANY
AMSTERDAM • NEW YORK • OXFORD

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1980

NORTH-HOLLAND PUBLISHING COMPANY – AMSTERDAM · NEW YORK · OXFORD

North-Holland ISBN: 0 444 70199 0

Published by:

North-Holland Publishing Company – Amsterdam · New York · Oxford

Sole distributors for the U.S.A. and Canada:

Elsevier North-Holland, Inc.
52 Vanderbilt Avenue
New York, N.Y. 10017

Library of Congress Cataloging in Publication Data

Moschovakis, Yiannis N.
Descriptive set theory.

(Studies in logic and the foundations of mathematics; v. 100)
Bibliography: p.
Includes index.
1. Descriptive set theory. I. Title. II Series,
QA248.M66 511'.3 79-11792
ISBN 0-444-85305-7

PRINTED IN THE NETHERLANDS

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This book is dedicated to the memory of my father Nicholas
a good and gentle man.

PREFACE

This book was conceived in the winter of 1970 when I heard that I was getting a Sloan Fellowship and I thought I would take a year off to write a book. It took a bit longer than that, but I have many good excuses.

I am grateful to the Sloan Foundation, the National Science Foundation and the University of California for their financial support—and to the Mathematics Department at UCLA for the stimulating and pleasant working environment that it provides.

One often sees in prefaces long lists of persons who have contributed to the project in one way or another and I hope I will be forgiven for not complying with tradition; in my case any reasonably complete list would have to start with Lebesgue and increase the size of the book beyond the publisher's indulgence. I will, however, mention my student Chris Freiling who read carefully through the entire final version of the manuscript and corrected all my errors.

My wife Joan is the only person who really knows how much I owe to her and she is too kind to tell. But I know too.

Finally, my deepest feelings of gratitude and appreciation are reserved for the very few friends with whom I have spent so many hours during the last ten years arguing about descriptive set theory; Bob Solovay and Tony Martin in the beginning, Aleko Kechris, Ken Kunen and Leo Harrington a little later. Their influence on my work will be obvious to anyone who glances through this book and I consider them my teachers—although of course, they are all so much younger than me. No doubt I would still work in this field if they were all priests or generals—but I would not enjoy it half as much.

Santa Monica, California December 22, 1978

Added in proof. I am deeply grateful to Dr. Haimanti Sarbadhikari who read the first seven chapters in proof and corrected all the errors missed by Chris Freiling. I am also indebted to Anna and Nicholas Moschovakis for their substantial help in constructing the indexes and to Tony Martin for the sustenance he offered me during the last stages of this work.

ABOUT THIS BOOK

My aim in this monograph is to give a brief but coherent exposition of the main results and methods of descriptive set theory. I have made no attempt to be complete; in a subject so broad this would degenerate into a long catalog of specialized results which would cover up the main thread. On the contrary, I have tried very hard to be selective, so that the central ideas stand out.

Much of the material is in the exercises. A very few of them are simple, to test the reader's comprehension and a few more give interesting extensions of the theory or sidelines. The vast majority of the exercises are an integral part of the monograph and would be normally billed "theorems." There are extensive "hints" for them, proofs really, with some of the details omitted.

I have tried hard to attribute all the important results and ideas to those who invented them but this was not an easy task and I have undoubtedly made many errors. *There is no suggestion that unattributed results are mine or are published here for the first time.* When I do not give credit for something, the most likely explanation is that I could not determine the correct credit. My own results are immodestly attributed to me, including those which are first published here.

Many of the references are in the historical sections at the end of each chapter. The paragraphs of these sections are numbered and the footnotes in the body of the text refer to these paragraphs—each time meaning the section at the end of the chapter where the reference occurs. In a first reading, it is best to skip looking up these references and read the historical sections as they come after one is familiar with the material in the chapter.

The order of exposition follows roughly the historical development of the subject, simply because this seemed the best way to do it. It goes without saying that the classical results are presented from a modern point of view and using modern notation.

What appeals to me most about descriptive set theory is that to study it you must really understand so many things: you need a little bit of topology, analysis and logic, a good deal of recursive function theory and a great deal of set theory, including constructibility, forcing, large cardinals and determinacy. What makes the writing of a book on the subject so difficult is that you must explain so many things: a little bit of topology, analysis and logic, a good deal of recursive function theory, etc. Of course, one could aim the book at those who already know all the prerequisites, but chances are that these few potential readers already know descriptive set theory. My aim has been to make this material accessible to a mathematician whose particular field of specialization could be anything, but who has an interest in set theory, or at least what used to be called "the theory of pointsets." He certainly knows whatever little topology and analysis are required, because he learned that as an undergraduate, and he has read Halmos' *Naive Set Theory* [1960] or a similar text. Beyond that, what he needs to read this book is patience and a basic interest in the central problem of descriptive set theory and definability theory in general: *to find and study the characteristic properties of definable objects.*

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INTRODUCTION

The roots of *Descriptive Set Theory* go back to the work of Borel, Baire and Lebesgue around the turn of the centrury, when the young French analysts were trying to come to grips with the abstract notion of a *function* introduced by Dirichlet and Riemann. A function was to be an arbitrary correspondence between objects, with no regard for any method or procedure by which this correspondence could be established. They had some doubts whether so general a concept should be accepted; in any case, it was obvious that all the specific functions which were studied in practice were determined by simple *analytic expressions*, explicit formulas, infinite series and the like. The problem was to delineate the functions which could be defined by such accepted methods and search for their characteristic properties, presumably nice properties not shared by all functions.

Baire was first to introduce in his Thesis [1899] what we now call *Baire functions* (of several real variables), the smallest set which contains all continuous functions and is closed under the taking of (pointwise) limits. He gave an inductive definition: the continuous functions are *of class* 0 and for each countable ordinal ξ, a function is *of class* ξ if it is the limit of a sequence of functions of smaller classes and is not itself of lower class. Baire, however, concentrated on a detailed study of the functions of class 1 and 2 and he said little about the general notion beyond the definition.

The first systematic study of definable functions was Lebesgue's [1905], *Sur les fonctions représentables analytiquement.* This beautiful and seminal paper truly started the subject of descriptive set theory.

Lebesgue defined the collection of *analytically representable functions as the smallest set which contains all constants and projections* $(x_1, x_2, \ldots, x_n) \mapsto x_i$ and which is closed under sums, products and the taking of limits. It is easy to verify that these are precisely the Baire functions. Lebesgue then showed that there exist Baire functions of every countable class and that there exist definable functions which are not analytically representable. He also defined the *Borel measurable* functions and showed that they too coincide with the Baire functions. In fact he proved a much stronger theorem along these lines which relates the *hierarchy* of Baire functions with a natural hierarchy of the Borel measurable sets at each level.

Today we recognize Lebesgue [1905] as a classic work in the *theory of definability*. It introduced and studied systematically several natural notions of definable functions and sets and it established the first important hierarchy theorems and structure results for collections of definable objects. In it we can find the origins of many standard tools and techniques that we use today, for example *universal sets* and applications of the Cantor *diagonal method* to questions of definability.

One of Lebesgue's results in [1905] identified the *implicity analytically definable* functions with the Baire functions. To take a simple case, suppose that $f:\mathcal{R}^2 \to \mathcal{R}$ is analytically representable and for each x,

$$f(x, y) = 0$$

has exactly one solution in y. This equation then defines y implicitly as a function of x; Lebesgue showed that it is an analytically representable function of x, by an argument which was "simple, short but false." The wrong step in the proof was hidden in a lemma taken as trivial, that a set in the line which is the projection of a Borel measurable set in the plane is itself Borel measurable.

Ten years later the error was spotted by Suslin, then a young student of Lusin at the University of Moscow, who rushed to tell his professor in a scene charmingly described in Sierpinski [1950].

Suslin called the projections of Borel sets *analytic* and showed that indeed there are analytic sets which are not Borel measurable. Together with Lusin they quickly established most of the basic properties of analytic sets and they announced their results in two short notes in the Comptes Rendus, Suslin [1917] and Lusin [1917].

The class of analytic sets is rich and complicated but the sets in it are nice. They are measurable in the sense of Lebesgue, they have the property of Baire and they satisfy the continuum hypothesis, i.e. every uncountable analytic set is equinumerous with the set of all real numbers. The best result in Suslin [1917] is a characterization of the Borel measurable sets as precisely those analytic sets which have analytic complements. Lusin [1917] announced another basic theorem which implied that Lebesgue's contention about implicitly analytically definable functions is true, despite the error in the original proof.

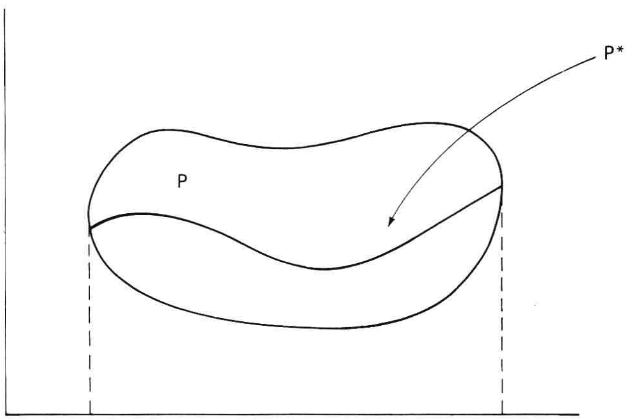
Suslin died in 1919 and the study of analytic sets was continued mostly by Lusin and his students in Moscow and by Sierpinski in Warsaw. Because of what Lusin delicately called "difficulties of international communication" those years, they were isolated from each other and from the wider mathematical community, and there were very few publications in western journals in the early twenties.

The next significant step was the introduction of *projective sets* by Lusin and Sierpinski in 1925: a set is projective if it can be constructed starting with Borel measurable sets and iterating the operations of projection and complementation. Using later terminology, let us call analytic sets A sets, analytic complements CA sets, projections of CA sets PCA sets, complements of these $CPCA$ sets, etc. Lusin in his [1925a], [1925b], [1925c] and Sierpinski [1925] showed that these classes of sets are all distinct and they established their elementary properties. But it was clear from the very beginning that the theory of projective sets was not easy. There was no obvious way to extend to these more complicated sets the regularity properties of Borel and analytic sets; for example, it was an open problem whether analytic complements satisfy the continuum hypothesis or whether PCA sets are Lebesgue measurable.

Another fundamental and difficult problem was posed in Lusin [1930a]. Suppose P is a subset of the plane; a subset P^* of P *uniformizes* P if P^* is the graph of a function and it has the same projection on the line as P.

The natural question is whether definable sets admit definable uniformizations and it comes up often, for example when we seek "canonical" solutions for y in terms of x in an equation

$$f(x, y) = 0.$$



Lusin and Sierpinski showed that Borel sets can be uniformized by analytic complements and Lusin also verified that analytic sets can be projectively uniformized. In a fundamental advance in the subject, Kondo [1938] completed earlier work of Novikov and proved that analytic complements and *PCA* sets can be uniformized by sets in the same classes. Again, there was no clear method for extending the known techniques to solve the uniformization problem for the higher projective classes.

As it turned out, the "difficulties of the theory of projective sets" which bothered Lusin from his very first publications in the subject could not be overcome by ingenuity alone. There was an insurmountable technical obstruction to answering the central open questions in the field, since *all of them were independent of the axioms of classical set theory.* It goes without saying that the researchers in descriptive set theory were formulating and trying to prove their assertions within axiomatic Zermelo–Fraenkel set theory, as all mathematicians still do, consciously or not.

The first independence results were proved by Gödel, in fact they were by-products of his famous consistency proof of the continuum hypothesis. He announced in his [1938] that in the model *L* of constructible sets there is a *PCA* set which is not Lebesgue measurable: it follows that one cannot establish in Zermelo–Frankel set theory (with the axiom of choice and even the continuum hypothesis) that all *PCA* sets are Lebesgue measurable. His results were followed up by some people, notably Mostowski and Kuratowski, but that was another period of "difficulties of international communication" and nothing was published until the late forties. Addison [1959b] gave the first exposition in print of the consistency and independence results that are obtained by analysing Gödel's *L.*

The independence of the continuum hypothesis was proved by Cohen [1963], whose powerful method of *forcing* was soon after applied to independence questions in descriptive set theory. One of the most significant papers in forcing was Solovay [1970], where it is shown (among other things) that one can consistently assume the axioms of Zermelo–Frankel set theory (with choice and even the continuum hypothesis) together with the proposition that all projective sets are Lebesgue measurable; from this and Gödel's work it follows that *in classical set theory we can neither prove nor disprove the Lebesgue measurability of PCA sets.*

Similar consistency and independence results were obtained about all the central problems left open in the classical period of descriptive set theory, say up to 1940. It says something about the power of the mathematicians working in the field those years, that in almost every instance they obtained the best theorems that could be proved from the axioms they were assuming.

So the logicians entered the picture in their usual style, as spoilers. There was, however, another parallel development which brought them in more substantially and in a friendlier role. Before going into that, let us make a few remarks about the appropriate context for studying problems of definability of functions and sets.

We have been recounting the development of descriptive set theory on the real numbers, but it is obvious that the basic notions are topological in nature and can be formulated in the context of more general topological spaces. All the important results can be extended easily to *complete, separable, metric spaces.* In fact, it was noticed early on that the theory assumes a particularly simple form on *Baire space*

$$\mathcal{N} = {}^{\omega}\omega,$$

the set of all infinite sequences of natural numbers, topologized with the product topology (taking ω discrete). The key fact about $\mathcal{N}$ is that it is homeomorphis with its own square $\mathcal{N} \times \mathcal{N}$, so that irrelevant problems of dimension do not come up. Results in the theory are often proved just for $\mathcal{N}$, with the (suitable) generalizations to other spaces and the reals in particular left for the reader or simply stated without proof.

Let us now go back to a discussion of the impact of logic and logicians on descriptive set theory.

The fundamental work of Gödel [1931] on incompleteness phenomena in formal systems suggested that it would be profitable to delineate and study those functions (of several variables) on the set ω of natural numbers which are *effectively computable.* A great deal of work was done on this problem in the thirties by Church, Kleene, Turing, Post and Gödel among others, from which emerged a coherent and beautiful theory of *computability* or *recursion.* The class of *recursive functions* (of several variables) *on* ω was characterized as the smallest set which contains all the constants, the successor and the projections $(x_1, x_2, \ldots, x_n) \mapsto x_i$ and which is closed under composition, a form of simple definition by induction (primitive recursion) and minimalization, where g is defined from f by the equation

$$g(x_1, x_2, \ldots, x_n) = \text{least } w \text{ such that } f(x_1, x_2, \ldots, x_n, w) = 0,$$