Mariano Giaquinta
Giuseppe Modica

Mathematical Analysis

Linear and Metric Structures and Continuity

Birkhäuser

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Boston • Basel • Berlin

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Cover design by Alex Gerasev.

Mathematics Subject Classification (2000): 00A35, 15-01, 32K99, 46L99, 32C18, 46E15, 46E20

Library of Congress Control Number: 2006927565

ISBN-10: 0-8176-4374-5
ISBN-13: 978-0-8176-4374-4
e-ISBN-10: 0-8176-4514-4
e-ISBN-13: 978-0-8176-4514-4

Printed on acid-free paper.

9 8 7 6 5 4 3 2 1

www.birkhauser.com

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Preface

One of the fundamental ideas of mathematical analysis is the notion of a *function*; we use it to describe and study relationships among variable quantities in a system and transformations of a system. We have already discussed real functions of one real variable and a few examples of functions of several variables[1], but there are many more examples of functions that the real world, physics, natural and social sciences, and mathematics have to offer:

(a) not only do we associate numbers and points to points, but we associate numbers or vectors to vectors,
(b) in the *calculus of variations* and in *mechanics* one associates an *energy* or *action* to each curve $y(t)$ connecting two points $(a, y(a))$ and $(b, y(b))$:

$$\Phi(y) := \int_a^b F(t, y(t), y'(t))\, dt$$

in terms of the so-called Lagrangian $F(t, y, p)$,
(c) in the theory of *integral equations* one maps a function into a new function

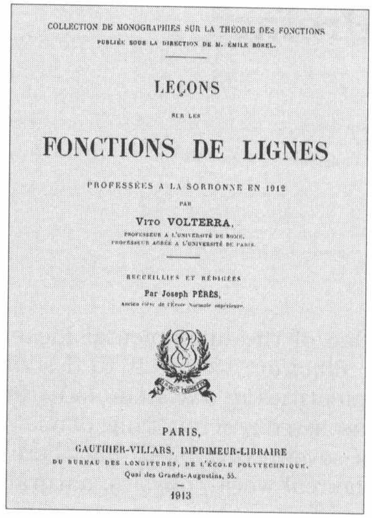
$$x(s) \quad\rightarrow\quad \int_a^b K(s, \tau) x(\tau)\, d\tau$$

by means of a *kernel* $K(s, \tau)$,
(d) in the theory of *differential equations* one considers transformations of a function $x(t)$ into the new function

$$t \;\rightarrow\; \int_a^t f(s, x(s))\, ds,$$

where $f(s, y)$ is given.

[1] in M. Giaquinta, G. Modica, *Mathematical Analysis. Functions of One Variable*, Birkhäuser, Boston, 2003, which we shall refer to as [GM1] and in M. Giaquinta, G. Modica, *Mathematical Analysis. Approximation and Discrete Processes*, Birkhäuser, Boston, 2004, which we shall refer to as [GM2].



COLLECTION DE MONOGRAPHIES SUR LA THÉORIE DES FONCTIONS
PUBLIÉE SOUS LA DIRECTION DE M. ÉMILE BOREL.

LEÇONS

SUR LES

FONCTIONS DE LIGNES

PROFESSÉES A LA SORBONNE EN 1912

PAR

VITO VOLTERRA,

PROFESSEUR A L'UNIVERSITÉ DE ROME,
PROFESSEUR AGRÉÉ A L'UNIVERSITÉ DE PARIS.

RECUEILLIES ET RÉDIGÉES

Par Joseph PÉRÈS,

Ancien élève de l'École Normale supérieure.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE.

Quai des Grands-Augustins, 55.

1913

Figure 0.1. Vito Volterra (1860–1940) and the frontispiece of his *Leçons sur les fonctions de lignes*.

Of course all the previous examples are covered by the abstract setting of functions or mappings from a set X (of numbers, points, functions, ...) with values in a set Y (of numbers, points, functions, ...). But in this general context we cannot grasp the richness and the specificity of the different situations, that is, the *essential* ingredients from the point of view of the question we want to study. In order to continue to treat these specificities in an abstract context in mathematics, but also use them in other fields, we proceed by identifying specific *structures* and studying the properties that only depend on these structures. In other words, we need to identify the relevant relationships among the elements of X and how these relationships reflect on the functions defined on X.

Of course we may define many intermediate structures. In this volume we restrict ourselves to illustrating some particularly important structures: that of a *linear* or *vector space* (the setting in which we may consider linear combinations), that of a *metric space* (in which we axiomate the notions of limit and continuity by means of a *distance*), that of a *normed vector space* (that combines linear and metric structures), that of a *Banach space* (where we may operate linearly and pass to the limit), and finally, that of a *Hilbert space* (that allows us to operate not only with the length of vectors, but also with the angles that they form).

The study of *spaces of functions* and, in particular, of *spaces of continuous functions* originating in Italy in the years 1870–1880 in the works of among others Vito Volterra (1860–1940), Giulio Ascoli (1843–1896), Cesare Arzelà (1847–1912) and Ulisse Dini (1845–1918), is especially relevant in the previous context.

A descriptive diagram is the following:

$$\begin{tikzcd}
& \boxed{\text{sets}} \arrow[dl] \arrow[dr] & \\
\boxed{\text{vector spaces}} \arrow[dr] & & \boxed{\text{metric spaces}} \arrow[dl] \\
& \boxed{\text{linear normed spaces}} \arrow[d] & \\
& \boxed{\text{Banach spaces}} \arrow[d] & \\
& \boxed{\text{Hilbert spaces}} &
\end{tikzcd}$$

Accordingly, this book is divided into three parts. In the first part we study the linear structure. In the first three chapters we discuss basic ideas and results, including Jordan's canonical form of matrices, and in the fourth chapter we present the spectral theorem for self-adjoint and normal operators in finite dimensions.

In the second part, we discuss the fundamental notions of general topology in the metric context in Chapters 5 and 6, continuous curves in Chapter 7, and finally, in Chapter 8 we illustrate the notions of homotopy and degree, and Brouwer's and Borsuk's theorems with a few applications to the topology of $\mathbb{R}^n$.

In the third part, after some basic preliminaries, we discuss in Chapter 9 the Banach space of continuous functions presenting some of the classical fixed point theorems that play a relevant role in the solvability of functional equations and, in particular, of differential equations. In Chapter 10 we deal with the theory of Hilbert spaces and the spectral theory of compact operators. Finally, in Chapter 9 we survey some of the important applications of the ideas and techniques that we previously developed to the study of geodesics, nonlinear ordinary differential and integral equations and trigonometric series.

In conclusion, this volume[2] aims at studying continuity and its implications both in finite- and infinite-dimensional spaces. It may be regarded as a companion to [GM1] and [GM2], and as a reference book for multi-dimensional calculus, since it presents the abstract context in which concrete problems posed by multi-dimensional calculus find their natural setting.

Though this volume discusses more advanced material than [GM1,2], we have tried to keep the same spirit, always providing examples and

[2] This book is a translation and revised edition of M. Giaquinta, G. Modica, *Analisi Matematica, III. Strutture lineari e metriche, continuità*, Pitagora Ed., Bologna, 2000.

exercises to clarify the main presentation, omitting several technicalities and developments that we thought to be too advanced and supplying the text with several illustrations.

We are greatly indebted to Cecilia Conti for her help in polishing our first draft and we warmly thank her. We would like to also thank Fabrizio Broglia and Roberto Conti for their comments when preparing the Italian edition; Laura Poggiolini, Marco Spadini and Umberto Tiberio for their comments and their invaluable help in catching errors and misprints and Stefan Hildebrandt for his comments and suggestions, especially those concerning the choice of illustrations. Our special thanks also go to all members of the editorial technical staff of Birkhäuser for the excellent quality of their work and especially to Avanti Paranjpye and the executive editor Ann Kostant.

Note: We have tried to avoid misprints and errors. But, as most authors, we are imperfect authors. We will be very grateful to anybody who wants to inform us about errors or just misprints or wants to express criticism or other comments. Our e-mail addresses are

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We shall try to keep up an errata corrige at the following webpages:

`http://www.sns.it/~giaquinta`

`http://www.dma.unifi.it/~modica`

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Pisa and Firenze
October 2006

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