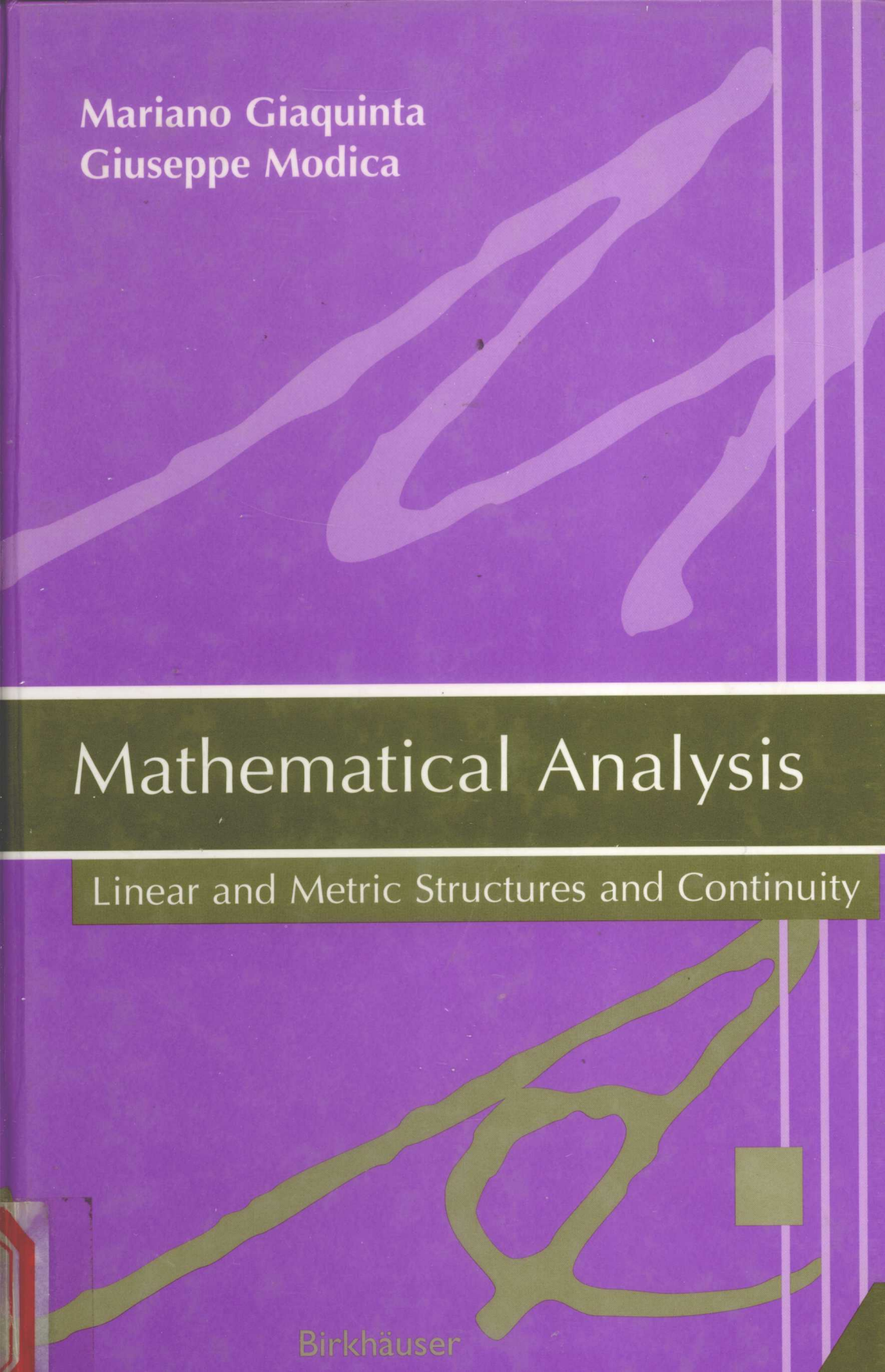




Mariano Giaquinta
Giuseppe Modica

Mathematical Analysis

Linear and Metric Structures and Continuity

Birkhäuser

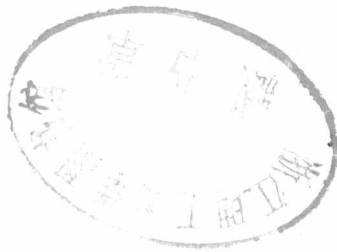


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Mathematical Analysis

*Linear and Metric Structures
and Continuity*



Birkhäuser
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Preface

One of the fundamental ideas of mathematical analysis is the notion of a *function*; we use it to describe and study relationships among variable quantities in a system and transformations of a system. We have already discussed real functions of one real variable and a few examples of functions of several variables¹, but there are many more examples of functions that the real world, physics, natural and social sciences, and mathematics have to offer:

- (a) not only do we associate numbers and points to points, but we associate numbers or vectors to vectors,
- (b) in the *calculus of variations* and in *mechanics* one associates an *energy* or *action* to each curve $y(t)$ connecting two points $(a, y(a))$ and $(b, y(b))$:

$$\Phi(y) := \int_a^b F(t, y(t), y'(t)) dt$$

in terms of the so-called Lagrangian $F(t, y, p)$,

- (c) in the theory of *integral equations* one maps a function into a new function

$$x(s) \rightarrow \int_a^b K(s, \tau)x(\tau) d\tau$$

by means of a *kernel* $K(s, \tau)$,

- (d) in the theory of *differential equations* one considers transformations of a function $x(t)$ into the new function

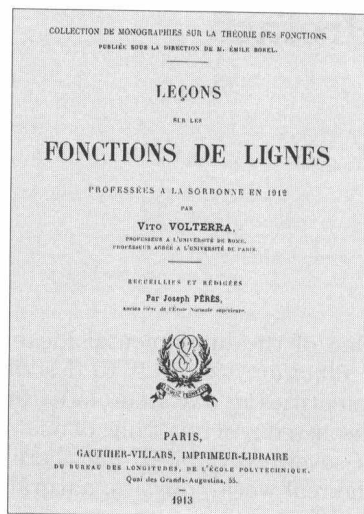
$$t \rightarrow \int_a^t f(s, x(s)) ds,$$

where $f(s, y)$ is given.

¹ in M. Giaquinta, G. Modica, *Mathematical Analysis. Functions of One Variable*, Birkhäuser, Boston, 2003, which we shall refer to as [GM1] and in M. Giaquinta, G. Modica, *Mathematical Analysis. Approximation and Discrete Processes*, Birkhäuser, Boston, 2004, which we shall refer to as [GM2].



Figure 0.1. Vito Volterra (1860–1940) and the frontispiece of his *Leçons sur les fonctions de lignes*.



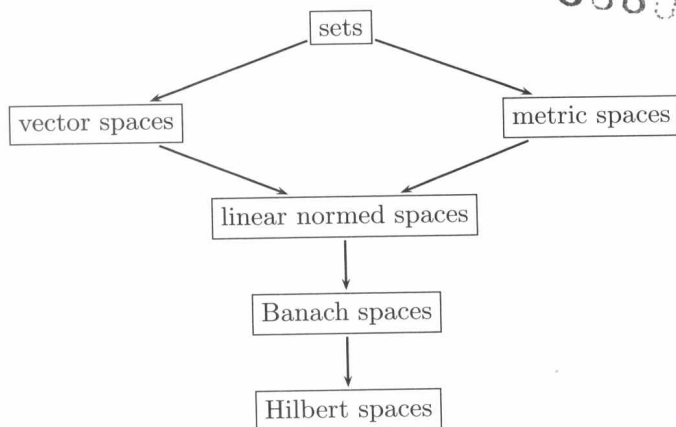
Of course all the previous examples are covered by the abstract setting of functions or mappings from a set X (of numbers, points, functions, ...) with values in a set Y (of numbers, points, functions, ...). But in this general context we cannot grasp the richness and the specificity of the different situations, that is, the *essential* ingredients from the point of view of the question we want to study. In order to continue to treat these specificities in an abstract context in mathematics, but also use them in other fields, we proceed by identifying specific *structures* and studying the properties that only depend on these structures. In other words, we need to identify the relevant relationships among the elements of X and how these relationships reflect on the functions defined on X .

Of course we may define many intermediate structures. In this volume we restrict ourselves to illustrating some particularly important structures: that of a *linear* or *vector space* (the setting in which we may consider linear combinations), that of a *metric space* (in which we axiomatize the notions of limit and continuity by means of a *distance*), that of a *normed vector space* (that combines linear and metric structures), that of a *Banach space* (where we may operate linearly and pass to the limit), and finally, that of a *Hilbert space* (that allows us to operate not only with the length of vectors, but also with the angles that they form).

The study of *spaces of functions* and, in particular, of *spaces of continuous functions* originating in Italy in the years 1870–1880 in the works of among others Vito Volterra (1860–1940), Giulio Ascoli (1843–1896), Cesare Arzelà (1847–1912) and Ulisse Dini (1845–1918), is especially relevant in the previous context.

A descriptive diagram is the following:

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Accordingly, this book is divided into three parts. In the first part we study the linear structure. In the first three chapters we discuss basic ideas and results, including Jordan's canonical form of matrices, and in the fourth chapter we present the spectral theorem for self-adjoint and normal operators in finite dimensions.

In the second part, we discuss the fundamental notions of general topology in the metric context in Chapters 5 and 6, continuous curves in Chapter 7, and finally, in Chapter 8 we illustrate the notions of homotopy and degree, and Brouwer's and Borsuk's theorems with a few applications to the topology of $\mathbb{R}^n$.

In the third part, after some basic preliminaries, we discuss in Chapter 9 the Banach space of continuous functions presenting some of the classical fixed point theorems that play a relevant role in the solvability of functional equations and, in particular, of differential equations. In Chapter 10 we deal with the theory of Hilbert spaces and the spectral theory of compact operators. Finally, in Chapter 9 we survey some of the important applications of the ideas and techniques that we previously developed to the study of geodesics, nonlinear ordinary differential and integral equations and trigonometric series.

In conclusion, this volume² aims at studying continuity and its implications both in finite- and infinite-dimensional spaces. It may be regarded as a companion to [GM1] and [GM2], and as a reference book for multi-dimensional calculus, since it presents the abstract context in which concrete problems posed by multi-dimensional calculus find their natural setting.

Though this volume discusses more advanced material than [GM1,2], we have tried to keep the same spirit, always providing examples and

² This book is a translation and revised edition of M. Giaquinta, G. Modica, *Analisi Matematica, III. Strutture lineari e metriche, continuità*, Pitagora Ed., Bologna, 2000.

exercises to clarify the main presentation, omitting several technicalities and developments that we thought to be too advanced and supplying the text with several illustrations.

We are greatly indebted to Cecilia Conti for her help in polishing our first draft and we warmly thank her. We would like to also thank Fabrizio Broglia and Roberto Conti for their comments when preparing the Italian edition; Laura Poggiolini, Marco Spadini and Umberto Tiberio for their comments and their invaluable help in catching errors and misprints and Stefan Hildebrandt for his comments and suggestions, especially those concerning the choice of illustrations. Our special thanks also go to all members of the editorial technical staff of Birkhäuser for the excellent quality of their work and especially to Avanti Paranjpye and the executive editor Ann Kostant.

Note: We have tried to avoid misprints and errors. But, as most authors, we are imperfect authors. We will be very grateful to anybody who wants to inform us about errors or just misprints or wants to express criticism or other comments. Our e-mail addresses are

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Pisa and Firenze
October 2006

Contents

Preface	v
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Part I. Linear Algebra

1. Vectors, Matrices and Linear Systems	3
1.1 The Linear Spaces $\mathbb{R}^n$ and $\mathbb{C}^n$	3
a. Linear combinations	3
b. Basis	6
c. Dimension	7
d. Ordered basis	9
1.2 Matrices and Linear Operators	10
a. The algebra of matrices	11
b. A few special matrices	12
c. Matrices and linear operators	13
d. Image and kernel	15
e. Grassmann's formula	18
f. Parametric and implicit equations of a subspace ..	18
1.3 Matrices and Linear Systems	22
a. Linear systems and the language of linear algebra	22
b. The Gauss elimination method	24
c. The Gauss elimination procedure for nonhomogeneous linear systems	29
1.4 Determinants	31
1.5 Exercises	37
2. Vector Spaces and Linear Maps	41
2.1 Vector Spaces and Linear Maps	41
a. Definition	41
b. Subspaces, linear combinations and bases	42
c. Linear maps	44
d. Coordinates in a finite-dimensional vector space ..	45
e. Matrices associated to a linear map	47
f. The space $\mathcal{L}(X, Y)$	49
g. Linear abstract equations	50

h.	Changing coordinates	51
i.	The associated matrix under changes of basis	53
j.	The dual space $\mathcal{L}(X, \mathbb{K})$	54
k.	The bidual space	55
l.	Adjoint or dual maps	56
2.2	Eigenvectors and Similar Matrices	57
2.2.1	Eigenvectors	58
a.	Eigenvectors and eigenvalues	58
b.	Similar matrices	60
c.	The characteristic polynomial	60
d.	Algebraic and geometric multiplicity	62
e.	Diagonalizable matrices	62
f.	Triangularizable matrices	64
2.2.2	Complex matrices	65
a.	The Cayley–Hamilton theorem	66
b.	Factorization and invariant subspaces	67
c.	Generalized eigenvectors and the spectral theorem	68
d.	Jordan’s canonical form	70
e.	Elementary divisors	75
2.3	Exercises	76
3.	Euclidean and Hermitian Spaces	79
3.1	The Geometry of Euclidean and Hermitian Spaces	79
a.	Euclidean spaces	79
b.	Hermitian spaces	82
c.	Orthonormal basis and the Gram–Schmidt algorithm	85
d.	Isometries	87
e.	The projection theorem	88
f.	Orthogonal subspaces	90
g.	Riesz’s theorem	91
h.	The adjoint operator	92
3.2	Metrics on Real Vector Spaces	95
a.	Bilinear forms and linear operators	95
b.	Symmetric bilinear forms or metrics	97
c.	Sylvester’s theorem	97
d.	Existence of g -orthogonal bases	99
e.	Congruent matrices	101
f.	Classification of real metrics	103
g.	Quadratic forms	104
h.	Reducing to a sum of squares	105
3.3	Exercises	109

4. Self-Adjoint Operators	111
4.1 Elements of Spectral Theory	111
4.1.1 Self-adjoint operators	111
a. Self-adjoint operators	111
b. The spectral theorem	112
c. Spectral resolution	114
d. Quadratic forms	115
e. Positive operators	117
f. The operators A^*A and AA^*	118
g. Powers of a self-adjoint operator	119
4.1.2 Normal operators	121
a. Simultaneous spectral decompositions	121
b. Normal operators on Hermitian spaces	121
c. Normal operators on Euclidean spaces	122
4.1.3 Some representation formulas	125
a. The operator A^*A	125
b. Singular value decomposition	126
c. The Moore–Penrose inverse	127
4.2 Some Applications	128
4.2.1 The method of least squares	128
a. The method of least squares	128
b. The function of linear regression	130
4.2.2 Trigonometric polynomials	130
a. Spectrum and products	131
b. Sampling of trigonometric polynomials	132
c. The discrete Fourier transform	134
4.2.3 Systems of difference equations	136
a. Systems of linear difference equations	136
b. Power of a matrix	137
4.2.4 An ODE system: small oscillations	141
4.3 Exercises	143

Part II. Metrics and Topology

5. Metric Spaces and Continuous Functions	149
5.1 Metric Spaces	151
5.1.1 Basic definitions	151
a. Metrics	151
b. Convergence	153
5.1.2 Examples of metric spaces	154
a. Metrics on finite-dimensional vector spaces	155
b. Metrics on spaces of sequences	157
c. Metrics on spaces of functions	159
5.1.3 Continuity and limits in metric spaces	161
a. Lipschitz-continuous maps between metric spaces	161
b. Continuous maps in metric spaces	162

c.	Limits in metric spaces	164
d.	The junction property	165
5.1.4	Functions from $\mathbb{R}^n$ into $\mathbb{R}^m$	166
a.	The vector space $C^0(A, \mathbb{R}^m)$	166
b.	Some nonlinear continuous transformations from $\mathbb{R}^n$ into $\mathbb{R}^N$	167
c.	The calculus of limits for functions of several variables	171
5.2	The Topology of Metric Spaces	174
5.2.1	Basic facts	175
a.	Open sets	175
b.	Closed sets	175
c.	Continuity	176
d.	Continuous real-valued maps	177
e.	The topology of a metric space	178
f.	Interior, exterior, adherent and boundary points ..	179
g.	Points of accumulation	180
h.	Subsets and relative topology	181
5.2.2	A digression on general topology	182
a.	Topological spaces	182
b.	Topologizing a set	184
c.	Separation properties	184
5.3	Completeness	185
a.	Complete metric spaces	185
b.	Completion of a metric space	186
c.	Equivalent metrics	187
d.	The nested sequence theorem	188
e.	Baire's theorem	188
5.4	Exercises	190
6.	Compactness and Connectedness	197
6.1	Compactness	197
6.1.1	Compact spaces	197
a.	Sequential compactness	197
b.	Compact sets in $\mathbb{R}^n$	198
c.	Coverings and ϵ -nets	199
6.1.2	Continuous functions and compactness	201
a.	The Weierstrass theorem	201
b.	Continuity and compactness	202
c.	Continuity of the inverse function	202
6.1.3	Semicontinuity and the Fréchet–Weierstrass theorem	203
6.2	Extending Continuous Functions	205
6.2.1	Uniformly continuous functions	205
6.2.2	Extending uniformly continuous functions to the closure of their domains	206
6.2.3	Extending continuous functions	207
a.	Lipschitz-continuous functions	207

6.2.4	Tietze's theorem	208
6.3	Connectedness	210
6.3.1	Connected spaces	210
	a. Connected subsets	211
	b. Connected components	211
	c. Segment-connected sets in $\mathbb{R}^n$	212
	d. Path-connectedness	213
6.3.2	Some applications	214
6.4	Exercises	216
7.	Curves	219
7.1	Curves in $\mathbb{R}^n$	219
7.1.1	Curves and trajectories	219
	a. The calculus	222
	b. Self-intersections	223
	c. Equivalent parametrizations	223
7.1.2	Regular curves and tangent vectors	224
	a. Regular curves	224
	b. Tangent vectors	225
	c. Length of a curve	226
	d. Arc length and C^1 -equivalence	232
7.1.3	Some celebrated curves	233
	a. Spirals	234
	b. Conchoids	236
	c. Cissoids	237
	d. Algebraic curves	238
	e. The cycloid	238
	f. The catenary	240
7.2	Curves in Metric Spaces	241
	a. Functions of bounded variation and rectifiable curves	241
	b. Lipschitz and intrinsic reparametrizations	243
7.2.1	Real functions with bounded variation	244
	a. The Cantor–Vitali function	245
7.3	Exercises	247
8.	Some Topics from the Topology of $\mathbb{R}^n$	249
8.1	Homotopy	250
8.1.1	Homotopy of maps and sets	250
	a. Homotopy of maps	250
	b. Homotopy classes	252
	c. Homotopy equivalence of sets	253
	d. Relative homotopy	256
8.1.2	Homotopy of loops	257
	a. The fundamental group with base point	257
	b. The group structure on $\pi_1(X, x_0)$	257
	c. Changing base point	258

	d. Invariance properties of the fundamental group . . .	259
8.1.3	Covering spaces	260
	a. Covering spaces	260
	b. Lifting of curves	261
	c. Universal coverings and homotopy	264
	d. A global invertibility result	264
8.1.4	A few examples	266
	a. The fundamental group of S^1	266
	b. The fundamental group of the figure eight	267
	c. The fundamental group of S^n , $n \geq 2$	267
8.1.5	Brouwer's degree	268
	a. The degree of maps $S^1 \rightarrow S^1$	268
	b. An integral formula for the degree	269
	c. Degree and inverse image	270
	d. The homological definition of degree for maps $S^1 \rightarrow S^1$	271
8.2	Some Results on the Topology of $\mathbb{R}^n$	272
8.2.1	Brouwer's theorem	272
	a. Brouwer's degree	272
	b. Extension of maps into S^n	273
	c. Brouwer's fixed point theorem	274
	d. Fixed points and solvability of equations in $\mathbb{R}^{n+1}$	275
	e. Fixed points and vector fields	276
8.2.2	Borsuk's theorem	278
8.2.3	Separation theorems	279
8.3	Exercises	281

Part III. Continuity in Infinite-Dimensional Spaces

9.	Spaces of Continuous Functions, Banach Spaces and Abstract Equations.	285
9.1	Linear Normed Spaces	285
9.1.1	Definitions and basic facts	285
	a. Norms induced by inner and Hermitian products	287
	b. Equivalent norms	288
	c. Series in normed spaces	288
	d. Finite-dimensional normed linear spaces	290
9.1.2	A few examples	292
	a. The space ℓ_p , $1 \leq p < \infty$	292
	b. A normed space that is not Banach	293
	c. Spaces of bounded functions	294
	d. The space $\ell_\infty(Y)$	295
9.2	Spaces of Bounded and Continuous Functions	295
9.2.1	Uniform convergence	295
	a. Uniform convergence	295
	b. Pointwise and uniform convergence	297

	c. A convergence diagram	297
	d. Uniform convergence on compact subsets	299
9.2.2	A compactness theorem	300
	a. Equicontinuous functions	300
	b. The Ascoli–Arzelà theorem	301
9.3	Approximation Theorems	303
9.3.1	Weierstrass and Bernstein theorems	303
	a. Weierstrass’s approximation theorem	303
	b. Bernstein’s polynomials	305
	c. Weierstrass’s approximation theorem for periodic functions	307
9.3.2	Convolutions and Dirac approximations	309
	a. Convolution product	309
	b. Mollifiers	312
	c. Approximation of the Dirac mass	313
9.3.3	The Stone–Weierstrass theorem	316
9.3.4	The Yosida regularization	319
	a. Baire’s approximation theorem	319
	b. Approximation in metric spaces	320
9.4	Linear Operators	322
9.4.1	Basic facts	322
	a. Continuous linear forms and hyperplanes	323
	b. The space of linear continuous maps	324
	c. Norms on matrices	324
	d. Pointwise and uniform convergence for operators	325
	e. The algebra $\text{End}(X)$	326
	f. The exponential of an operator	327
9.4.2	Fundamental theorems	327
	a. The principle of uniform boundedness	328
	b. The open mapping theorem	329
	c. The closed graph theorem	330
	d. The Hahn–Banach theorem	331
9.5	Some General Principles for Solving Abstract Equations	334
9.5.1	The Banach fixed point theorem	335
	a. The fixed point theorem	335
	b. The continuity method	337
9.5.2	The Caccioppoli–Schauder fixed point theorem	339
	a. Compact maps	339
	b. The Caccioppoli–Schauder theorem	341
	c. The Leray–Schauder principle	342
9.5.3	The method of super- and sub-solutions	342
	a. Ordered Banach spaces	343
	b. Fixed points via sub- and super-solutions	344
9.6	Exercises	344

10. Hilbert Spaces, Dirichlet's Principle and Linear Compact Operators	351
10.1 Hilbert Spaces	351
10.1.1 Basic facts	351
a. Definitions and examples	351
b. Orthogonality	354
10.1.2 Separable Hilbert spaces and basis	355
a. Complete systems and basis	355
b. Separable Hilbert spaces	355
c. Fourier series and ℓ_2	357
d. Some orthonormal polynomials in L^2	360
10.2 The Abstract Dirichlet's Principle and Orthogonality	363
a. The abstract Dirichlet's principle	364
b. Riesz's theorem	366
c. The orthogonal projection theorem	367
d. Projection operators	368
10.3 Bilinear Forms	368
10.3.1 Linear operators and bilinear forms	369
a. Linear operators	369
b. Adjoint operator	369
c. Bilinear forms	370
10.3.2 Coercive symmetric bilinear forms	371
a. Inner products	371
b. Green's operator	372
c. Ritz's method	373
d. Linear regression	374
10.3.3 Coercive nonsymmetric bilinear forms	376
a. The Lax–Milgram theorem	376
b. Faedo–Galerkin method	377
10.4 Linear Compact Operators	378
10.4.1 Fredholm–Riesz–Schauder theory	378
a. Linear compact operators	378
b. The alternative theorem	379
c. Some facts related to the alternative theorem	381
d. The alternative theorem in Banach spaces	383
e. The spectrum of compact operators	384
10.4.2 Compact self-adjoint operators	385
a. Self-adjoint operators	385
b. Spectral theorem	387
c. Compact normal operators	388
d. The Courant–Hilbert–Schmidt theory	390
e. Variational characterization of eigenvalues	392
10.5 Exercises	393

11. Some Applications	395
11.1 Two Minimum Problems	395
11.1.1 Minimal geodesics in metric spaces	395
a. Semicontinuity of the length	395
b. Compactness	396
c. Existence of minimal geodesics	397
11.1.2 A minimum problem in a Hilbert space	397
a. Weak convergence in Hilbert spaces	398
b. Existence of minimizers of convex coercive functionals	400
11.2 A Theorem by Gelfand and Kolmogorov	402
11.3 Ordinary Differential Equations	403
11.3.1 The Cauchy problem	404
a. Velocities of class $C^k(D)$	404
b. Local existence and uniqueness	405
c. Continuation of solutions	407
d. Systems of higher order equations	409
e. Linear systems	410
f. A direct approach to Cauchy problem for linear systems	411
g. Continuous dependence on data	413
h. The Peano theorem	415
11.3.2 Boundary value problems	416
a. The shooting method	418
b. A maximum principle	419
c. The method of super- and sub-solutions	421
d. A theorem by Bernstein	423
11.4 Linear Integral Equations	424
11.4.1 Some motivations	424
a. Integral form of second order equations	425
b. Materials with memory	425
c. Boundary value problems	426
d. Equilibrium of an elastic thread	427
e. Dynamics of an elastic thread	427
11.4.2 Volterra integral equations	429
11.4.3 Fredholm integral equations in C^0	430
11.5 Fourier's Series	431
11.5.1 Definitions and preliminaries	433
a. Dirichlet's kernel	435
11.5.2 Pointwise convergence	436
a. The Riemann–Lebesgue theorem	436
b. Regular functions and Dini test	437
11.5.3 L^2 -convergence and the energy equality	439
a. Fourier's partial sums and orthogonality	439
b. A first uniform convergence result	440
c. Energy equality	441
11.5.4 Uniform convergence	442

a. A variant of the Riemann–Lebesgue theorem	442
b. Uniform convergence for Dini-continuous functions	444
c. Riemann’s localization principles	445
11.5.5 A few complementary facts	445
a. The primitive of the Dirichlet kernel	445
b. Gibbs’s phenomenon	447
11.5.6 The Dirichlet–Jordan theorem	449
a. The Dirichlet–Jordan test	449
b. Féjer example	451
11.5.7 Féjer’s sums	452
A. Mathematicians and Other Scientists	455
B. Bibliographical Notes	457
C. Index	459