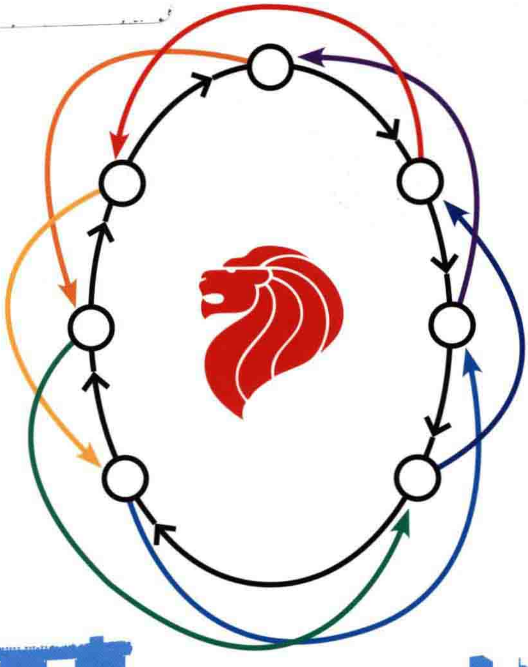
Graph Theory

Undergraduate Mathematics



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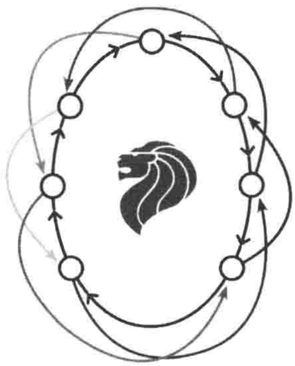

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The digraph on the cover, sometimes referred to as the Koh-digraph, is the smallest counterexample to a conjecture by Lovász. The Lion Head Symbol is inserted in the center of the digraph to commemorate the 50th anniversary of independence of Singapore in 2015.

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Preface

Discrete Mathematics is a branch of mathematics dealing with finite or countable processes and elements. Graph Theory is an area in Discrete Mathematics which studies configurations involving a set of vertices interconnected by edges (called graphs). From humble beginnings and almost recreational type problems, Graph Theory has found its calling in the modern world of complex systems and especially of the computer. Graph Theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science, operational research, management sciences and the life sciences. Since computers require discrete formulation of problems, Graph Theory has become an essential and powerful tool for engineers and applied scientists, in particular, in the area of designing and analyzing algorithms for various problems which range from designing the itineraries for a shipping company to sequencing the human genome in the life sciences.

This book is an expansion of our first book Introduction to Graph Theory: H3 Mathematics. While the first book was intended for capable high school students and university freshmen, this version covers substantially more ground and is intended as a reference and textbook for undergraduate studies in Graph Theory. In fact, the topics cover a few modules in the Graph Theory taught at the National University of Singapore. The material in the book, and especially the variety and quantity of the problems, are derived very much from the enormous wealth of knowledge and experience gained from the thirty plus years of teaching and researching of the first author, Koh Khee Meng.

Certain features of this book are worth mentioning. Care is specially taken so that concepts are explained clearly and developed properly; it strives to be readable and at the same time be mathematically rigorous.

At suitable junctures, questions are inserted for discussion. This is to ensure that the reader understands the preceding section fully before proceeding on to new ideas and concepts. Each chapter includes applications of the concepts in real-life. They are added for general interest and as substantiation of the usefulness of Graph Theory concepts. There are many items in the Exercise component following most sections. Some are exercises intended for reinforcing what is learnt earlier while others test the full range of understanding and problem solving in the concepts acquired. Each chapter concludes with a large selection of interesting problems that cover all the sections in that chapter. Some of these problems are from research articles and these add to the depth and cutting-edge aspect of the endeavor. Proofs of most important theorems are given in their full mathematical rigour. Finally, we believe that "good research enlivens teaching, and good teaching encourages research" and so we have made a conscious effort to include recent research work at the frontiers of areas of graph theory into this book.

Chapter 1 covers the fundamental concepts and basic results in Graph Theory tracing its history from Euler's solution of the problem of the Bridges of Konigsberg. Fundamental concepts include those of graphs, multigraphs, vertex degrees, paths, cycles, distance, eccentricity of vertices, and the radius and diameter of a graph.

In Chapter 2, congruence is defined in terms of isomorphism rather than a vague notion of shape and this allows a 'handle' to compare graphs. Chapter 2 further fleshes out the concept of a graph by introducing the attendant concepts of subgraphs and the complement of a graph, and coding of a graph via graphic sequences.

In Chapter 3, we introduce two important families of graphs, namely trees and bipartite graphs. A tree, in some sense, forms the 'skeleton' of a connected graph and in general, a forest of trees forms the 'skeleton' of any graph. Thus, the structure and properties of trees are very important. Bipartite graphs are another family of graphs that have found applications in many real-life situations such as matching a group of job seekers with a set of potential jobs under certain conditions.

Chapter 4 returns the reader to Euler's seminal work on the Bridges of Konigsberg. Euler is memorialized for his contribution by having graphs with the property that one can have a walk that traverses all edges exactly once and then returns to the starting vertex named after him – Eulerian multigraphs. This chapter gives a fuller treatment of Eulerian multigraphs. This study includes the Chinese Postman Problem which is an optimization problem on multigraphs with weighted edges.

William Rowan Hamilton was a famous mathematical prodigy. He introduced the vertex analogous concept of the Eulerian circuit. Such a graph that admits a cycle that visits all the vertices in the graph was named Hamiltonian in his honor. In Chapter 5, we study Hamiltonian graphs and the Traveling Salesman Problem which is another important optimization problem on graphs with weighted edges.

Be it transportation systems, computer networks, or supply chains, the notion of the robustness of a system, in the sense of the connectedness of its components, is certainly of great importance. Since graphs are used often to model systems, in Chapter 6, we shall introduce an important parameter, called connectivity, which measures how 'strong' the connectedness of a given graph is. Two versions of the notion, namely, the vertex version and the edge version, will be presented.

Three important features of sets of vertices and sets of edges are discussed in Chapter 7. The first arises from the 'natural' relation of independence among vertices, giving rise to the notion of an independent set, intuitively seen as a set of vertices with no edges between them. Another 'natural' relation of independence among edges gives rise to the notion of a matching, intuitively seen as a set of edges with no common vertex between any pair of them. Thirdly, the analogous notions of a vertex cover and an edge cover are intuitively seen as a set of vertices which 'touch' every edge in the graph and a set of edges which 'touch' every vertex in the graph. Independence, matching and covering are graph features that are useful when modeling some real-life scenarios. For example, a complete matching can be used to find a system of distinct representatives (SDR). And in the next chapter, independence is a necessary condition for vertex-coloring, which in turn can be used in scheduling activities.

Can all maps be colored with at most four colors? Many people believed that the answer is affirmative, but no one could prove it for a long time. This is known as the Four Color Problem. In Chapter 8, we study vertex-colorings of graphs as an approach to tackle this problem. The chromatic number is introduced and an algorithm and some techniques to estimate or enumerate it are discussed. Interesting applications of vertex-coloring to scheduling problems are given in some detail.

A set of vertices in a graph is called a dominating set if any vertex not in the set is adjacent to a vertex in the set. This idea of dominating sets in graphs can be used to solve real world problems. For example, suppose transmitting stations are to be built in some cities in a country so that every city without a transmitting station can receive messages from some adjacent city with one. The problem for selecting the cities for the transmitting stations is that of finding a dominating set in the corresponding graph. Chapter 9 studies this very useful concept and its variants, including the very recent conception of Roman Domination.

Finally, Chapter 10 studies graphs with 'directions' indicated on the edges. Such are called directed graphs or digraphs. Such digraphs suitably model many situations where relationships between items are directional. The chapter covers some basic concepts and provides some detail for a special family of digraphs, called tournaments. Material in this chapter also reaches as far as the frontier area of optimal orientations of graphs.

Problems in a section are referenced as

Problem [Chapter].[Section].[Number];

for example, Problem 1.3.4 means Problem 4 at the end of Section 1.3. General problems related to all the concepts in a chapter are placed in a special section at the end of the chapter and are referenced as Problem [Chapter].[Number]; for example, Problem IV.3 means Problem 3 at the end of Chapter 4. While a list of notations is provided at the beginning of the book, a list of indices can be found at the end. References are cited in the book according to the format, for example, Dirac (1960) refers to the article authored by Dirac published in 1960.

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Koh Khee Meng
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Ng Kah Loon
Tay Eng Guan

Notations

Symbol	Meaning	Page
$V, V(G)$	vertex set of G	6
$E, E(G)$	edge set of G	6
$v(G)$	order of G	6
$e(G)$	size of G	7
$N(v)$	set of all neighbors of v in G	14
$N[v]$	the union of $N(v)$ and $\{v\}$	368
$N(S)$	set of vertices in G which are adjacent to some vertex in S	269
$d(v)$	degree of v	15
O_n	empty graph (or null graph) of order n	21
K_n	complete graph of order n	22
P_n	path of order n	109
C_n	cycle of order n	23
F_n	fan of order n	163
W_n	wheel of order n	163
Q_n	n-cube graph	46
$K(p, q)$	complete bipartite graph having p and q vertices in the two partite sets	104
$K(p_1, p_2, \cdots, p_r)$	complete r-partite graph having $p_1, p_2, \cdots, p_r$ vertices in the r partite sets	160
$\lceil x \rceil$	smallest integer greater than or equal to x	48
$\lfloor x \rfloor$	largest integer smaller than or equal to x	159
$d(u, v)$	distance from u to v	38
$e(v)$	eccentricity of v	40
$\mathrm{rad}(G)$	radius of G	41

$\operatorname{diam}(G)$	diameter of G	41
$C(G)$	center of G	41
$c(G)$	number of components in G	32
$cl(G)$	closure of G	217
$g(G)$	girth of G	91
$w(H)$	weight of a graph H	144
$w(f)$	weight of a function f	383
$o(G)$	number of odd components of G	286
$\dim(G)$	metric dimension of G	95
$G \cong H$	G is isomorphic to H	50
$G - F$	subgraph of G obtained by deleting the edges in F from G	61
$G - W$	subgraph of G obtained by removing vertices in W from $V(G)$ together with all edges incident with them from $E(G)$	65
$[W]$	subgraph induced by a set of vertices W	64
$[F]$	subgraph induced by a set of edges F	145
$n_G(R)$	number of subgraphs of G which are isomorphic to R	63
$e(A, V(G)\backslash A)$	number of edges having one end in A and the other in $V(G)\backslash A$	74
$G \circ e$, where $e = uv$	multigraph obtained from G by first deleting all edges joining u and v, and then identifying u and v	131
$pn\,[w, S]$	set of all private neighbors of w with respect to S	368
$\boldsymbol{A}(G)$	adjacency matrix of G	9
$\boldsymbol{A}^k(G)$	kth power of $\boldsymbol{A}(G)$	34
$\boldsymbol{B}(G)$	incidence matrix of G	11
$\boldsymbol{B}(G)^T$	transpose of $\boldsymbol{B}(G)$	25
$\boldsymbol{D}(G)$	degree matrix of G	137
$\overline{G}$	complement of G	75
G^k	kth power of G	91
$L(G)$	line graph of G	92
$G + H$	join of G and H	88
$G \,\Box\, H$	cartesian product of G and H	93

$G \wedge H$	the graph given by $(G-wg) \cup (H-wh)+gh$ where w is a common vertex of G and H, wg is an edge in G and wh is an edge in H	336
$\Delta(G)$	maximum degree of G	19
$\delta(G)$	minimum degree of G	19
$\tau(G)$	number of spanning trees of G	128
$\kappa(G)$	vertex-connectivity of G	239
$\kappa'(G)$	edge-connectivity of G	240
$\rho(G)$	deficiency of G	275
$\alpha(G)$	independence number of G	293
$\alpha'(G)$	matching number of G	293
$\beta(G)$	vertex covering number of G	294
$\beta'(G)$	edge covering number of G	296
$\chi(G)$	chromatic number of G	313
$\omega(G)$	clique number of G	317
$\gamma(G)$	domination number of G	373
$i(G)$	independent domination number of G	378
$\gamma_R(G)$	Roman domination number of G	384
$i_R(G)$	independent Roman domination number of G	388
$d^-(v)$	in-degree of v	398
$d^+(v)$	out-degree of v	398
$s(v)$	score of a vertex v	416
$I(v)$	in-set of v	423
$O(v)$	out-set of v	423
$G(D)$	underlying graph of digraph D	401
$\text{diam}(D)$	diameter of digraph D	403
$A(D)$	adjacency matrix of digraph D	411
$\overrightarrow{D}$	converse of digraph D	414
$K_r(D)$	set of all r-kings in digraph D	436
$k_r(D)$	number of r-kings in digraph D	436
$\overrightarrow{d}(G)$	orientation number of G	451
T_n	tournament of order n	416

Contents