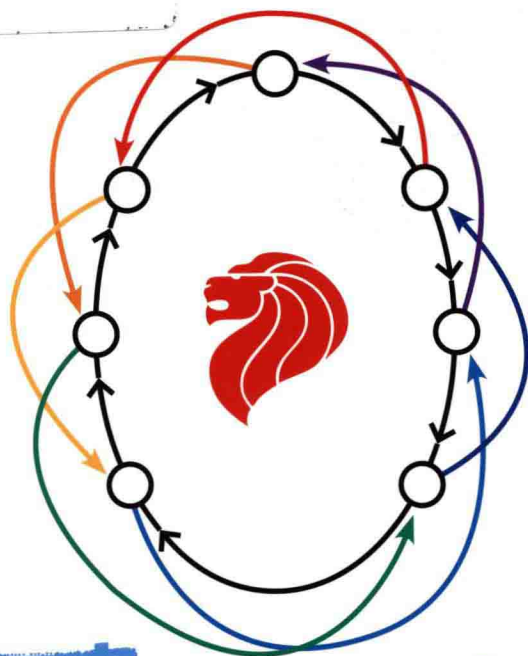


Graph Theory

Undergraduate Mathematics



Khee Meng Koh • Fengming Dong

Kah Loon Ng • Eng Guan Tay

Graph Theory

Undergraduate Mathematics



National University of Singapore, Singapore

Fengming Dong

Nanyang Technological University, Singapore

Kah Loon Ng

National University of Singapore, Singapore

Eng Guan Tay

Nanyang Technological University, Singapore

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

Library of Congress Cataloging-in-Publication Data

Koh, K. M. (Khee Meng), 1944–

Graph theory : undergraduate mathematics / by Khee Meng Koh (NUS, Singapore), Fengming Dong (NTU, Singapore), Kah Loon Ng (NUS, Singapore), Eng Guan Tay (NTU, Singapore).

pages cm

NUS represents National University of Singapore. NTU represents Nanyang Technological University, Singapore.

Includes bibliographical references and index.

ISBN 978-9814641586 (hardcover : alk. paper) -- ISBN 978-9814641593 (pbk. : alk. paper)

1. Graph theory--Textbooks. 2. Mathematics--Study and teaching (Higher) I. Dong, Fengming 1962-- II. Ng, Kah Loon. III. Tay, Eng Guan. IV. Title.

QA166.K635 2015

511'.5--dc23

2014044827

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

The digraph on the cover, sometimes referred to as the Koh-digraph, is the smallest counterexample to a conjecture by Lovász. The Lion Head Symbol is inserted in the center of the digraph to commemorate the 50th anniversary of independence of Singapore in 2015.

Copyright © 2015 by World Scientific Publishing Co. Pte. Ltd.

All rights reserved. This book, or parts thereof, may not be reproduced in any form or by any means, electronic or mechanical, including photocopying, recording or any information storage and retrieval system now known or to be invented, without written permission from the publisher.

For photocopying of material in this volume, please pay a copying fee through the Copyright Clearance Center, Inc., 222 Rosewood Drive, Danvers, MA 01923, USA. In this case permission to photocopy is not required from the publisher.

Printed in Singapore

Graph Theory

Undergraduate Mathematics

Preface

Discrete Mathematics is a branch of mathematics dealing with finite or countable processes and elements. Graph Theory is an area in Discrete Mathematics which studies configurations involving a set of vertices interconnected by edges (called graphs). From humble beginnings and almost recreational type problems, Graph Theory has found its calling in the modern world of complex systems and especially of the computer. Graph Theory and its applications can be found not only in other branches of mathematics, but also in scientific disciplines such as engineering, computer science, operational research, management sciences and the life sciences. Since computers require discrete formulation of problems, Graph Theory has become an essential and powerful tool for engineers and applied scientists, in particular, in the area of designing and analyzing algorithms for various problems which range from designing the itineraries for a shipping company to sequencing the human genome in the life sciences.

This book is an expansion of our first book *Introduction to Graph Theory: H3 Mathematics*. While the first book was intended for capable high school students and university freshmen, this version covers substantially more ground and is intended as a reference and textbook for undergraduate studies in Graph Theory. In fact, the topics cover a few modules in the Graph Theory taught at the National University of Singapore. The material in the book, and especially the variety and quantity of the problems, are derived very much from the enormous wealth of knowledge and experience gained from the thirty plus years of teaching and researching of the first author, Koh Khee Meng.

Certain features of this book are worth mentioning. Care is specially taken so that concepts are explained clearly and developed properly; it strives to be readable and at the same time be mathematically rigorous.

At suitable junctures, questions are inserted for discussion. This is to ensure that the reader understands the preceding section fully before proceeding on to new ideas and concepts. Each chapter includes applications of the concepts in real-life. They are added for general interest and as substantiation of the usefulness of Graph Theory concepts. There are many items in the Exercise component following most sections. Some are exercises intended for reinforcing what is learnt earlier while others test the full range of understanding and problem solving in the concepts acquired. Each chapter concludes with a large selection of interesting problems that cover all the sections in that chapter. Some of these problems are from research articles and these add to the depth and cutting-edge aspect of the endeavor. Proofs of most important theorems are given in their full mathematical rigour. Finally, we believe that “good research enlivens teaching, and good teaching encourages research” and so we have made a conscious effort to include recent research work at the frontiers of areas of graph theory into this book.

Chapter 1 covers the fundamental concepts and basic results in Graph Theory tracing its history from Euler’s solution of the problem of the Bridges of Königsberg. Fundamental concepts include those of graphs, multigraphs, vertex degrees, paths, cycles, distance, eccentricity of vertices, and the radius and diameter of a graph.

In Chapter 2, congruence is defined in terms of isomorphism rather than a vague notion of shape and this allows a ‘handle’ to compare graphs. Chapter 2 further fleshes out the concept of a graph by introducing the attendant concepts of subgraphs and the complement of a graph, and coding of a graph via graphic sequences.

In Chapter 3, we introduce two important families of graphs, namely trees and bipartite graphs. A tree, in some sense, forms the ‘skeleton’ of a connected graph and in general, a forest of trees forms the ‘skeleton’ of any graph. Thus, the structure and properties of trees are very important. Bipartite graphs are another family of graphs that have found applications in many real-life situations such as matching a group of job seekers with a set of potential jobs under certain conditions.

Chapter 4 returns the reader to Euler’s seminal work on the Bridges of Königsberg. Euler is memorialized for his contribution by having graphs with the property that one can have a walk that traverses all edges exactly once and then returns to the starting vertex named after him – Eulerian multigraphs. This chapter gives a fuller treatment of Eulerian multigraphs. This study includes the Chinese Postman Problem which is an optimization

problem on multigraphs with weighted edges.

William Rowan Hamilton was a famous mathematical prodigy. He introduced the vertex analogous concept of the Eulerian circuit. Such a graph that admits a cycle that visits all the vertices in the graph was named Hamiltonian in his honor. In Chapter 5, we study Hamiltonian graphs and the Traveling Salesman Problem which is another important optimization problem on graphs with weighted edges.

Be it transportation systems, computer networks, or supply chains, the notion of the robustness of a system, in the sense of the connectedness of its components, is certainly of great importance. Since graphs are used often to model systems, in Chapter 6, we shall introduce an important parameter, called connectivity, which measures how ‘strong’ the connectedness of a given graph is. Two versions of the notion, namely, the vertex version and the edge version, will be presented.

Three important features of sets of vertices and sets of edges are discussed in Chapter 7. The first arises from the ‘natural’ relation of independence among vertices, giving rise to the notion of an independent set, intuitively seen as a set of vertices with no edges between them. Another ‘natural’ relation of independence among edges gives rise to the notion of a matching, intuitively seen as a set of edges with no common vertex between any pair of them. Thirdly, the analogous notions of a vertex cover and an edge cover are intuitively seen as a set of vertices which ‘touch’ every edge in the graph and a set of edges which ‘touch’ every vertex in the graph. Independence, matching and covering are graph features that are useful when modeling some real-life scenarios. For example, a complete matching can be used to find a system of distinct representatives (SDR). And in the next chapter, independence is a necessary condition for vertex-coloring, which in turn can be used in scheduling activities.

Can all maps be colored with at most four colors? Many people believed that the answer is affirmative, but no one could prove it for a long time. This is known as the Four Color Problem. In Chapter 8, we study vertex-colorings of graphs as an approach to tackle this problem. The chromatic number is introduced and an algorithm and some techniques to estimate or enumerate it are discussed. Interesting applications of vertex-coloring to scheduling problems are given in some detail.

A set of vertices in a graph is called a dominating set if any vertex not in the set is adjacent to a vertex in the set. This idea of dominating sets in graphs can be used to solve real world problems. For example, suppose transmitting stations are to be built in some cities in a country so that every

city without a transmitting station can receive messages from some adjacent city with one. The problem for selecting the cities for the transmitting stations is that of finding a dominating set in the corresponding graph. Chapter 9 studies this very useful concept and its variants, including the very recent conception of Roman Domination.

Finally, Chapter 10 studies graphs with ‘directions’ indicated on the edges. Such are called directed graphs or digraphs. Such digraphs suitably model many situations where relationships between items are directional. The chapter covers some basic concepts and provides some detail for a special family of digraphs, called tournaments. Material in this chapter also reaches as far as the frontier area of optimal orientations of graphs.

Problems in a section are referenced as

Problem [Chapter].[Section].[Number];

for example, Problem 1.3.4 means Problem 4 at the end of Section 1.3. General problems related to all the concepts in a chapter are placed in a special section at the end of the chapter and are referenced as Problem [Chapter].[Number]; for example, Problem IV.3 means Problem 3 at the end of Chapter 4. While a list of notations is provided at the beginning of the book, a list of indices can be found at the end. References are cited in the book according to the format, for example, Dirac (1960) refers to the article authored by Dirac published in 1960.

We would like to thank Goh Chee Ying, Hang Kim Hoo, Ku Cheng Yeaw, Zeinab Maleki, Ng Boon Leong, Soh Kian Wee, Tay Tiong Seng, Chia Gek Ling, Tan Ban Pin, Ting Tao Siang and Anders Yeo for reading through the draft and checking through the problems – any mistakes that remain are ours alone.

*Koh Khee Meng
Dong Fengming
Ng Kah Loon
Tay Eng Guan*

Notations

Symbol	Meaning	Page
$V, V(G)$	vertex set of G	6
$E, E(G)$	edge set of G	6
$v(G)$	order of G	6
$e(G)$	size of G	7
$N(v)$	set of all neighbors of v in G	14
$N[v]$	the union of $N(v)$ and $\{v\}$	368
$N(S)$	set of vertices in G which are adjacent to some vertex in S	269
$d(v)$	degree of v	15
O_n	empty graph (or null graph) of order n	21
K_n	complete graph of order n	22
P_n	path of order n	109
C_n	cycle of order n	23
F_n	fan of order n	163
W_n	wheel of order n	163
Q_n	n -cube graph	46
$K(p, q)$	complete bipartite graph having p and q vertices in the two partite sets	104
$K(p_1, p_2, \dots, p_r)$	complete r -partite graph having p_1, p_2, \dots, p_r vertices in the r partite sets	160
$\lceil x \rceil$	smallest integer greater than or equal to x	48
$\lfloor x \rfloor$	largest integer smaller than or equal to x	159
$d(u, v)$	distance from u to v	38
$e(v)$	eccentricity of v	40
$\text{rad}(G)$	radius of G	41

$\text{diam}(G)$	diameter of G	41
$C(G)$	center of G	41
$c(G)$	number of components in G	32
$cl(G)$	closure of G	217
$g(G)$	girth of G	91
$w(H)$	weight of a graph H	144
$w(f)$	weight of a function f	383
$o(G)$	number of odd components of G	286
$\dim(G)$	metric dimension of G	95
$G \cong H$	G is isomorphic to H	50
$G - F$	subgraph of G obtained by deleting the edges in F from G	61
$G - W$	subgraph of G obtained by removing vertices in W from $V(G)$ together with all edges incident with them from $E(G)$	65
$[W]$	subgraph induced by a set of vertices W	64
$[F]$	subgraph induced by a set of edges F	145
$n_G(R)$	number of subgraphs of G which are isomorphic to R	63
$e(A, V(G) \setminus A)$	number of edges having one end in A and the other in $V(G) \setminus A$	74
$G \circ e$, where $e = uv$	multigraph obtained from G by first deleting all edges joining u and v , and then identifying u and v	131
$pn[w, S]$	set of all private neighbors of w with respect to S	368
$A(G)$	adjacency matrix of G	9
$A^k(G)$	k th power of $A(G)$	34
$B(G)$	incidence matrix of G	11
$B(G)^T$	transpose of $B(G)$	25
$D(G)$	degree matrix of G	137
\overline{G}	complement of G	75
G^k	k th power of G	91
$L(G)$	line graph of G	92
$G + H$	join of G and H	88
$G \square H$	cartesian product of G and H	93

$G \wedge H$	the graph given by $(G - wg) \cup (H - wh) + gh$ where w is a common vertex of G and H , wg is an edge in G and wh is an edge in H	336
$\Delta(G)$	maximum degree of G	19
$\delta(G)$	minimum degree of G	19
$\tau(G)$	number of spanning trees of G	128
$\kappa(G)$	vertex-connectivity of G	239
$\kappa'(G)$	edge-connectivity of G	240
$\rho(G)$	deficiency of G	275
$\alpha(G)$	independence number of G	293
$\alpha'(G)$	matching number of G	293
$\beta(G)$	vertex covering number of G	294
$\beta'(G)$	edge covering number of G	296
$\chi(G)$	chromatic number of G	313
$\omega(G)$	clique number of G	317
$\gamma(G)$	domination number of G	373
$i(G)$	independent domination number of G	378
$\gamma_R(G)$	Roman domination number of G	384
$i_R(G)$	independent Roman domination number of G	388
$d^-(v)$	in-degree of v	398
$d^+(v)$	out-degree of v	398
$s(v)$	score of a vertex v	416
$I(v)$	in-set of v	423
$O(v)$	out-set of v	423
$G(D)$	underlying graph of digraph D	401
$\text{diam}(D)$	diameter of digraph D	403
$A(D)$	adjacency matrix of digraph D	411
\overrightarrow{D}	converse of digraph D	414
$K_r(D)$	set of all r -kings in digraph D	436
$k_r(D)$	number of r -kings in digraph D	436
$\overrightarrow{d}(G)$	orientation number of G	451
T_n	tournament of order n	416

Contents

<i>Preface</i>	v
<i>Notations</i>	ix
1. Fundamental Concepts and Basic Results	1
1.1 The Königsberg Bridge Problem	1
1.2 Multigraphs and Graphs	2
Exercise for Section 1.2	12
1.3 Vertex Degree	14
Exercise for Section 1.3	25
1.4 Paths, Cycles and Connectedness	26
Exercise for Section 1.4	34
1.5 Distance between Two Vertices	37
Exercise for Section 1.5	44
1.6 Problem Set I	46
2. Graph Isomorphisms, Subgraphs, the Complement of a Graph and Graphic Sequences	49
2.1 Isomorphic Graphs and Isomorphisms	49
2.2 Testing Isomorphic Graphs	53
Exercise for Section 2.2	57
2.3 Subgraphs of a Graph	60
Exercise for Section 2.3	68
2.4 The Complement of a Graph	74
Exercise for Section 2.4	78
2.5 Graphic Sequences	80
Exercise for Section 2.5	87

2.6	Problem Set II	90
3.	Bipartite Graphs and Trees	97
3.1	Bipartite Graphs	97
	Exercise for Section 3.1	106
3.2	Trees	108
	Exercise for Section 3.2	114
3.3	Spanning Trees of a Graph	116
	Exercise for Section 3.3	126
3.4	The Number of Spanning Trees	128
	Exercise for Section 3.4	135
3.5	Kirchhoff's Matrix-Tree Theorem and Cayley's Formula	136
	Exercise for Section 3.5	142
3.6	Two Problems on Weighted Graphs	143
	Exercise for Section 3.6	153
3.7	Problem Set III	156
4.	Eulerian Multigraphs and The Chinese Postman Problem	165
4.1	Euler Circuits and Eulerian Multigraphs	165
	Exercise for Section 4.1	167
4.2	Characterizations of Eulerian Multigraphs	168
4.3	Semi-Eulerian Multigraphs	174
	Exercise for Section 4.3	178
4.4	Finding Euler Circuits and Trails	182
	Exercise for Section 4.4	186
4.5	The Chinese Postman Problem	187
	Exercise for Section 4.5	195
4.6	Problem Set IV	197
5.	Hamiltonian Graphs and The Traveling Salesman Problem	201
5.1	Around the World and Spanning Cycles of a Graph	201
5.2	A Necessary Condition for a Graph to be Hamiltonian	207
	Exercise for Section 5.2	210
5.3	Sufficient Conditions for a Graph to be Hamiltonian	214
	Exercise for Section 5.3	219
5.4	The Traveling Salesman Problem	221
	Exercise for Section 5.4	230
5.5	Problem Set V	232

6.	Connectivity	237
6.1	Motivation	237
6.2	Vertex-connectivity and Edge-connectivity	238
	Exercise for Section 6.2	240
6.3	The Triple $(\kappa, \kappa', \delta)$	242
	Exercise for Section 6.3	244
6.4	k -connected Graphs	245
	Exercise for Section 6.4	251
6.5	Problem Set VI	255
7.	Independence, Matching and Covering	259
7.1	Introduction	259
7.2	Matchings in Bipartite Graphs	260
	Exercise for Section 7.2	266
7.3	Hall's Theorem	269
	Exercise for Section 7.3	272
7.4	System of Distinct Representatives	277
	Exercise for Section 7.4	280
7.5	Maximum Matchings and Perfect Matchings	282
	Exercise for Section 7.5	291
7.6	Independence, Covering and Gallai Identities	293
	Exercise for Section 7.6	298
7.7	Problem Set VII	301
8.	Vertex-colorings and Planar Graphs	309
8.1	The Four-color Problem	309
8.2	Vertex-colorings and Chromatic Number	310
	Exercise for Section 8.2	315
8.3	Computation of Chromatic Number	316
	Exercise for Section 8.3	319
8.4	Greedy Coloring Algorithm	323
	Exercise for Section 8.4	327
8.5	Brooks' Theorem	330
	Exercise for Section 8.5	334
8.6	Critical Graphs	334
	Exercise for Section 8.6	340
8.7	Planar Graphs and the Four-color Theorem	341
	Exercise for Section 8.7	346

8.8	Applications	347
	Exercise for Section 8.8	351
8.9	Problem Set VIII	355
9.	Domination	363
9.1	Introduction	363
9.2	Dominating Sets and Minimal Dominating Sets	366
	Exercise for Section 9.2	370
9.3	Domination Number	372
	Exercise for Section 9.3	376
9.4	Independent Domination	377
	Exercise for Section 9.4	380
9.5	Roman Domination	382
	Exercise for Section 9.5	386
9.6	Problem Set IX	387
10.	Digraphs and Tournaments	393
10.1	Digraphs	393
	Exercise for Section 10.1	396
10.2	Basic Concepts	397
	Exercise for Section 10.2	408
10.3	Tournaments	416
	Exercise for Section 10.3	419
10.4	Paths and Cycles in Tournaments	421
	Exercise for Section 10.4	426
10.5	The Score Sequence of a Tournament	427
	Exercise for Section 10.5	434
10.6	Kings and Bases	435
	Exercise for Section 10.6	445
10.7	Optimal Orientations of Graphs	448
	Exercise for Section 10.7	457
10.8	Problem Set X	459
	<i>Bibliography</i>	465
	<i>Index</i>	475