Mathematical Game Theory and Applications

Vladimir Mazalov

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Mathematical Game Theory and Applications

Preface

This book offers a combined course of lectures on game theory which the author has delivered for several years in Russian and foreign universities.

In addition to classical branches of game theory, our analysis covers modern branches left without consideration in most textbooks on the subject (negotiation models, potential games, parlor games, best choice games, and network games). The fundamentals of mathematical analysis, algebra, and probability theory are the necessary prerequisites for reading.

The book can be useful for students specializing in applied mathematics and informatics, as well as economical cybernetics. Moreover, it attracts the mutual interest of mathematicians operating in the field of game theory and experts in the fields of economics, management science, and operations research.

Each chapter concludes with a series of exercises intended for better understanding. Some exercises represent open problems for conducting independent investigations. As a matter of fact, stimulation of reader's research is the main priority of the book. A comprehensive bibliography will guide the audience in an appropriate scientific direction.

For many years, the author has enjoyed the opportunity to discuss derived results with Russian colleagues L.A. Petrosjan, V.V. Zakharov, N.V. Zenkevich, I.A. Seregin, and A.Yu. Garnaev (St. Petersburg State University), A.A. Vasin (Lomonosov Moscow State University), D.A. Novikov (Trapeznikov Institute of Control Sciences, Russian Academy of Sciences), A.V. Kryazhimskii and A.B. Zhizhchenko (Steklov Mathematical Institute, Russian Academy of Sciences), as well as with foreign colleagues M. Sakaguchi (Osaka University), M. Tamaki (Aichi University), K. Szajowski (Wroclaw University of Technology), B. Monien (University of Paderborn), K. Avratchenkov (INRIA, Sophia-Antipolis), and N. Perrin (University of Lausanne). They all have my deep and sincere appreciation. The author expresses profound gratitude to young colleagues A.N. Rettieva, J.S. Tokareva, Yu.V. Chirkova, A.A. Ivashko, A.V. Shiptsova and A.Y. Kondratjev from Institute of Applied Mathematical Research (Karelian Research Center, Russian Academy of Sciences) for their assistance in typing and formatting of the book. Next, my frank acknowledgement belongs to A.Yu. Mazurov for his careful translation, permanent feedback, and contribution to the English version of the book.

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Introduction

"*Equilibrium arises from righteousness, and righteousness arises from the meaning of the cosmos.*"

From Hermann Hesse's *The Glass Bead Game*

Game theory represents a branch of mathematics, which analyzes models of optimal decision-making in the conditions of a conflict. Game theory belongs to operations research, a science originally intended for planning and conducting military operations. However, the range of its applications appears much wider. Game theory always concentrates on models with several participants. This forms a fundamental distinction of game theory from optimization theory. Here the notion of an optimal solution is a matter of principle. There exist many definitions of the solution of a game. Generally, the solution of a game is called an equilibrium, but one can choose different concepts of an equilibrium (a Nash equilibrium, a Stackelberg equilibrium, a Wardrop equilibrium, to name a few).

In the last few years, a series of outstanding researchers in the field of game theory were awarded Nobel Prize in Economic Sciences. They are J.C. Harsanyi, J.F. Nash Jr., and R. Selten (1994) "for their pioneering analysis of equilibria in the theory of non-cooperative games," F.E. Kydland and E.C. Prescott (2004) "for their contributions to dynamic macroeconomics: the time consistency of economic policy and the driving forces behind business cycles," R.J. Aumann and T.C. Schelling (2005) "for having enhanced our understanding of conflict and cooperation through game-theory analysis," L. Hurwicz, E.S. Maskin, and R.B. Myerson (2007) "for having laid the foundations of mechanism design theory." Throughout the book, we will repeatedly cite these names and corresponding problems.

Depending on the number of players, one can distinguish between zero-sum games (antagonistic games) and nonzero-sum games. Strategy sets are finite or infinite (matrix games and games on compact sets, respectively). Next, players may act independently or form coalitions; the corresponding models represent non-cooperative games and cooperative games. There are games with complete or partial incoming information.

Game theory admits numerous applications. One would hardly find a field of sciences focused on life and society without usage of game-theoretic methods. In the first place, it is necessary to mention economic models, models of market relations and competition, pricing models, models of seller-buyer relations, negotiation, and stable agreements, etc. The pioneering book by J. von Neumann and O. Morgenstern, the founders of game theory, was entitled *Theory of Games and Economic Behavior*. The behavior of market participants, modeling of their psychological features forms the subject of a new science known as experimental economics.

Game-theoretic methods generated fundamental results in evolutionary biology. The notion of evolutionary stable strategies introduced by British biologist J.M. Smith enabled explaining the evolution of several behavioral peculiarities of animals such as aggressiveness, migration, and struggle for survival. Game-theoretic methods are intensively used in rational nature management problems. For instance, fishing quotas distribution in the ocean, timber extraction by several participants, agricultural pricing are problems of game theory. Today, it seems even impossible to implement intergovernmental agreements on natural resources utilization and environmental pollution reduction (e.g., The Kyoto Protocol) without game-theoretic analysis. In political sciences, game theory concerns voting models in parliaments, influence assessment models for certain political factions, as well as models of defense resources distribution for stable peace achievement. In jurisprudence, game theory is applied in arbitration for assessing the behavioral impact of conflicting sides on judicial decisions.

We have recently observed a technological breakthrough in the analysis of the virtual information world. In terms of game theory, all participants of the global computer network (Internet) and mobile communication networks represent interacting players that receive and transmit information by appropriate data channels. Each player pursues individual interests (acquire some information or complicate this process). Players strive for channels with high-level capacities, and the problem of channel distribution among numerous players arises naturally. And game-theoretic methods are of assistance here. Another problem concerns the impact of user service centralization on system efficiency. The estimate of the centralization effect in a system, where each participant follows individual interests (maximal channel capacity, minimal delay, the maximal amount of received information, etc.) is known as the price of anarchy. Finally, an important problem lies in defining the influence of information network topology on the efficiency of player service. These are non-trivial problems causing certain paradoxes. We describe the corresponding phenomena in the book.

Which fields of knowledge manage without game-theoretic methods? Perhaps, medical science and finance do so, although game-theoretic methods have also recently found some applications in these fields.

The approach to material presentation in this book differs from conventional ones. We intentionally avoid a detailed treatment of matrix games, as far as they are described in many publications. Our study begins with nonzero-sum games and the fundamental theorem on equilibrium existence in convex games. Later on, this result is extended to the class of zero-sum games. The discussion covers several classical models used in economics (the models of market competition suggested by Cournot, Bertrand, Hotelling, and Stackelberg, as well as auctions). Next, we pass from normal-form games to extensive-form games and parlor games. The early chapters of the book consider two-player games, and further analysis embraces n-player games (first, non-cooperative games, and then cooperative ones).

Subsequently, we provide fundamental results in new branches of game theory, best choice games, network games, and dynamic games. The book proposes new schemes of negotiations, much attention is paid to arbitration procedures. Some results belong to the author and his colleagues. The fundamentals of mathematical analysis, algebra, and probability theory are the necessary prerequisites for reading.

This book contains an accompanying website. Please visit www.wiley.com/go/game_theory.

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1

Strategic-form two-player games

Introduction

Our analysis of game problems begins with the case of two-player strategic-form (equivalently, normal-form) games. The basic notions of game theory comprise **Players, Strategies and Payoffs**. In the sequel, denote players by I and II. A normal-form game is organized in the following way. Player I chooses a certain strategy x from a set X, while player II simultaneously chooses some strategy y from a set Y. In fact, the sets X and Y may possess any structure (a finite set of values, a subset of R^n, a set of measurable functions, etc.). As a result, players I and II obtain the payoffs $H_1(x, y)$ and $H_2(x, y)$, respectively.

Definition 1.1 *A normal-form game is an object*

$$\Gamma =< I, II, X, Y, H_1, H_2 >,$$

where X, Y designate the sets of strategies of players I and II, whereas H_1, H_2 indicate their payoff functions, $H_i : X \times Y \to R, i = 1, 2$.

Each player selects his strategy regardless of the opponent's choice and strives for maximizing his own payoff. However, a player's payoff depends both on his strategy and the behavior of the opponent. This aspect makes the specifics of game theory.

How should one comprehend the solution of a game? There exist several approaches to construct solutions in game theory. Some of them will be discussed below. First, let us consider the notion of a Nash equilibrium as a central concept in game theory.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

Definition 1.2 *A Nash equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *meeting the conditions*

$$H_1(x, y^*) \leq H_1(x^*, y^*),$$
$$H_2(x^*, y) \leq H_2(x^*, y^*) \qquad (1.1)$$

for arbitrary strategies x, y *of the players.*

Inequalities (1.1) imply that, as the players deviate from a Nash equilibrium, their payoffs do decrease. Hence, deviations from the equilibrium appear non-beneficial to any player. Interestingly, there may exist no Nash equilibria. Therefore, a major issue in game problems concerns their existence. Suppose that a Nash equilibrium exists; in this case, we say that the payoffs $H_1^* = H_1(x^*, y^*)$, $H_2^* = H_2(x^*, y^*)$ are optimal. A set of strategies (x, y) is often called a **strategy profile**.

1.1 The Cournot duopoly

We mention the Cournot duopoly [1838] among pioneering game models that gained wide popularity in economic research. The term "duopoly" corresponds to a two-player game.

Imagine two companies, I and II, manufacturing some quantities of a same product (q_1 and q_2, respectively). In this model, the quantities represent the strategies of the players. The market price of the product equals an initial price p after deduction of the total quantity $Q = q_1 + q_2$. And so, the unit price constitutes $(p - Q)$. Let c be the unit cost such that $c < p$. Consequently, the players' payoffs take the form

$$H_1(q_1, q_2) = (p - q_1 - q_2)q_1 - cq_1, \;\; H_2(q_1, q_2) = (p - q_1 - q_2)q_2 - cq_2. \qquad (1.2)$$

In the current notation, the game is defined by $\Gamma =< I, II, Q_1 = [0, \infty), Q_2 = [0, \infty), H_1, H_2 >$.

Nash equilibrium evaluation (see formula (1.1)) calls for solving two problems, *viz.*, $\max_{q_1} H_1(q_1, q_2^*)$ and $\max_{q_2} H_2(q_1^*, q_2)$. Moreover, we have to demonstrate that the maxima are attained at $q_1 = q_1^*$, $q_2 = q_2^*$. The quadratic functions $H_1(q_1, q_2^*)$ and $H_2(q_1^*, q_2)$ get maximized by

$$q_1 = \frac{1}{2}\left(p - c - q_2^*\right)$$
$$q_2 = \frac{1}{2}\left(p - c - q_1^*\right).$$

Naturally, these quantities must be non-negative, which dictates that

$$q_i^* \leq p - c, \;\; i = 1, 2. \qquad (1.3)$$

By resolving the derived system of equations in q_1^*, q_2^*, we find

$$q_1^* = q_2^* = \frac{p - c}{3}$$