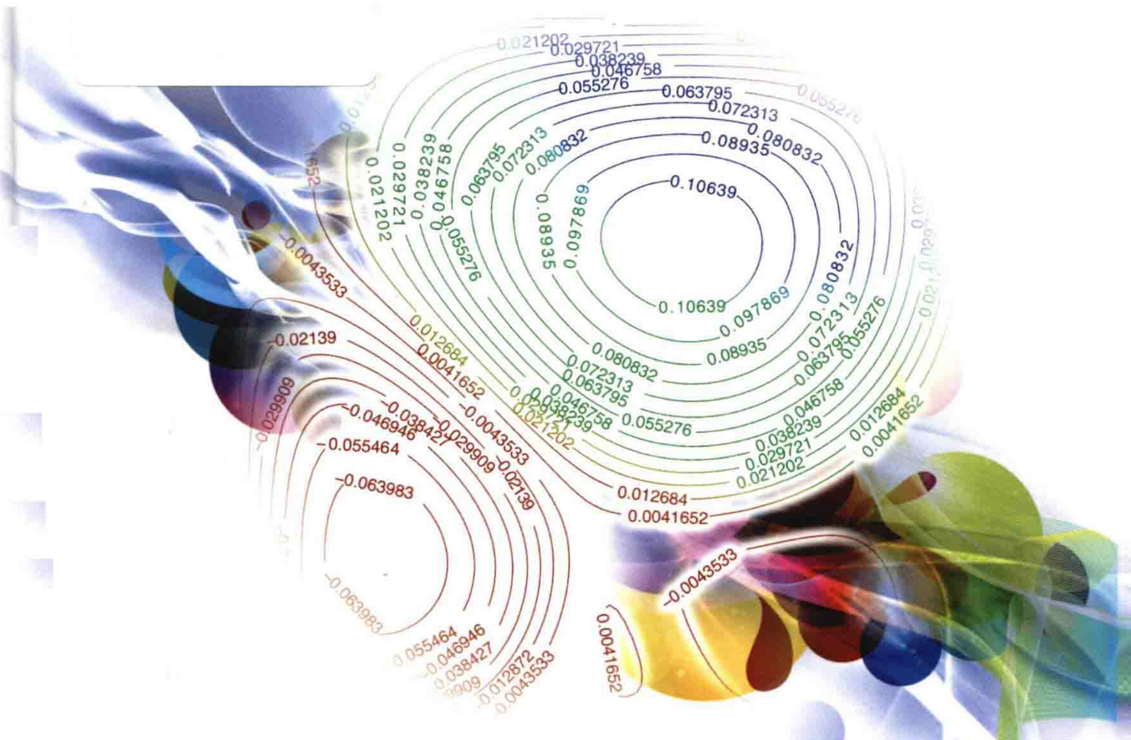
NAVIER–STOKES EQUATIONS IN PLANAR DOMAINS



Matania Ben-Artzi

Jean-Pierre Croisille

Dalia Fishelov

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Matania Ben-Artzi
Hebrew University of Jerusalem, Israel

Jean-Pierre Croisille
Université de Lorraine, France

Dalia Fishelov
Afeka Tel-Aviv Academic College of Engineering, Israel

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Preface

This monograph is devoted to the Navier–Stokes system in planar domains.

The Navier–Stokes system of equations, modeling incompressible, viscous flow in two or three space dimensions, is one of the most well-known classical systems of fluid dynamics and, in fact, of mathematical physics in general.

The pioneering work of Leray in the 1930s established the global (in time) well-posedness of the system in the two-dimensional case. It seemed therefore that this case was "closed." However, this was certainly not the case; the subsequent development of fluid dynamics in three directions, theoretical, numerical and experimental, led to a great interest in flow problems associated with "singular objects." These objects, including point vortices or vortex filaments, have been well-known since the early days of fluid dynamics in the eighteenth century. Their evolution in time serves as a central feature in the overall description of the flow.

A fact that is common to all these singular cases is that the L^2 norm of the associated velocity field (corresponding to the total energy of the fluid) is not finite. On the other hand, the basic premise of Leray's theory is the assumption that this norm is finite! Thus, there has been no rigorous mathematical theory applicable, say, to the motion of point vortices in two-dimensional flows, while the numerical treatment of such motion ("vortex methods") has increased in popularity since the 1970s.

The rigorous treatment of the two-dimensional Navier–Stokes system with such "rough initial data" was taken up only in the second half of the 1980s. The first part of this monograph gives a detailed exposition of these developments, based on a classical parabolic approach. It is based on the *vorticity formulation* of the system. A fundamental role is played by the integral operators associated with the heat kernel and the Biot–Savart kernel (relating the vorticity to the velocity).

The system is considered either in the whole plane $\mathbb{R}^2$ or in a square with periodic boundary conditions. Thus, physical boundary conditions (such as the "no-slip" condition) are not considered in this part. It is remarkable that the rigorous treatment of the motion of an initial vortex in a bounded planar domain remains an open problem!

On the numerical side, on the other hand, we cannot avoid the case of a bounded domain, subject to physically relevant boundary conditions. However, such boundary conditions are not readily translated into "vorticity boundary conditions." The progress made in the last twenty years in terms of "compact schemes" has produced very efficient algorithms for the approximation of the biharmonic operator. In turn, the use of the *stream-function formulation* of the system has become a very attractive option; the system is reduced to a scalar equation and the boundary conditions are naturally implemented. The appearance of the biharmonic operator in this equation seems a reasonable price to pay. The second part of this monograph gives a detailed account of this approach, based primarily on the authors' work during the last decade.

We provide detailed introductions to the two parts, where background material is expounded. The first (theoretical) part is supplemented with an Appendix containing topics from functional analysis that are not readily found in basic books on partial differential equations. Thus, this part should be accessible not only to specialists in mathematical analysis, but also to graduate students and researchers in the physical sciences who are interested in the rigorous theory.

In the second (numerical) part, we have made an effort to make it wholly self-contained. The basic relevant facts concerning difference operators are expounded, making the passage to modern compact schemes accessible even to readers having no background in numerical analysis. The accuracy of the schemes as well as the convergence of discrete solutions to the continuous ones are discussed in detail.

This monograph grew out of talks, joint papers and short courses given by us at our respective institutes and elsewhere. Stimulating discussions with S. Abarbanel, C. Bardos, B. Bialecki, A. Chorin, A. Ditkowski, A. Ern, G. Fairweather, S. Friedlander, T. Gallay, J. Gibbon, R. Glowinski, J.-L. Guermond, G. Katriel, P. Minev, M. Schonbek, C.-W. Shu, R. Temam, S. Trachtenberg, E. Turkel, and D. Ye were very valuable to us.

One of us (D.F.) was the first graduate student of the late David Gottlieb. All three of us enjoyed his hospitality at Brown University during the summer of 2007. In spite of his failing health, he generously gave us his attention and his comments helped us shape Part II of the monograph.

The second author (J.-P.C.) thanks B. Courbet, D. Dutoya, J. Falcovitz, and F. Haider for illuminating discussions on the numerical treatment and the physical understanding of fluid flows.

Special thanks are due to I. Chorev. Section 11.2 is based on his M.Sc. thesis written under the supervision of M.B-A.

Our joint work demanded frequent trips between France and Israel. Support for these trips was provided through the French–Israeli scientific cooperation: we gratefully acknowledge the Arc-en-Ciel/Keshet program and the French-Israel High Council for Scientific and Technological Cooperation (French Ministry of Foreign Affairs, French Ministry of Research and Israeli Ministry of Science and Technology). We also acknowledge the support of the Hebrew University of Jerusalem, the University Paul Verlaine-Metz and the Afeka Tel Aviv Academic College of Engineering.

This work could not have been realized without the support of our families.

Jerusalem, Metz, Tel Aviv, August 2011

Contents

PART I
Basic Theory

Big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity.

L. F. Richardson, *Weather Prediction by Numerical Process* (1922).

Chapter 1

Introduction

In this part of the book we establish the existence, uniqueness and regularity theory of solutions to the Navier–Stokes equations in two spatial dimensions. In addition, we also discuss the large-time asymptotic behavior of these solutions.

The main theme of the monograph (in both the theoretical and the numerical parts) is the **evolution of the vorticity** in the planar geometric setting. The recognition of the vorticity as a central object in the understanding of fluid flow dates back to the early days of this field. We refer the reader to the classic books [123, 124], where the physical significance, as well as numerous examples, are expounded.

More recently, a growing number of researchers, both in *Mathematical Fluid Dynamics* and in *Computational Fluid Dynamics*, have turned their attention to the vorticity in their studies. This increased interest is well reflected in the books [43, 133, 157].

The present monograph also highlights the fundamental role of *vorticity* (and its associated *streamfunction*) in the study of the Navier–Stokes system. However, the topics treated here are different from those addressed in the aforementioned books [43,133,157]. Indeed, Chorin's book is mostly devoted to the statistical physics aspects of vorticity (and turbulence), which we do not discuss here. The book by Majda and Bertozzi presents a unified treatment of the Navier–Stokes and the Euler equations, focusing primarily on the latter, whereas we deal exclusively with the Navier–Stokes equations in two dimensions. Our aim is to study the evolution of vorticity for rather general initial data, beyond the classical Leray theory. We shall return to this later in this chapter.

We refer to [44, 123, 124] for extensive discussion of the basic equations governing viscous (incompressible) fluid flow. We briefly recall these equations in the physical three-dimensional setting, and then restrict to the two-dimensional case, the subject matter of this monograph.

In order to distinguish between scalar and vector-valued functions we use boldface for vectors and vector functions in $\mathbb{R}^n$. Their components are labeled as $\mathbf{w} = (w^1, \ldots, w^n)$. In particular, for the Navier–Stokes equations this convention applies both to the planar case ($n = 2$) and the three-dimensional case. The scalar product is denoted by $\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^n a^i \cdot b^i$ and the Euclidean norm is $|\mathbf{w}|^2 = \sum_{i=1}^n (w^i)^2$.

Partial derivatives with respect to the time or spatial coordinates are denoted, respectively, by $\partial_t = \frac{\partial}{\partial t}$, $\partial_{x^i} = \frac{\partial}{\partial x^i}$. Using the gradient operator $\boldsymbol{\nabla} = \{\partial_{x^1}, ..., \partial_{x^n}\}$ we can represent the divergence and curl of a vector field $\mathbf{u}$ by, respectively, $\boldsymbol{\nabla} \cdot \mathbf{u}$ and $\boldsymbol{\nabla} \times \mathbf{u}$.

Occasionally (especially in integration) we will write $\boldsymbol{\nabla}_{\mathbf{x}}$ for clarity.

The Laplacian operator is $\Delta = \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} = \sum_{i=1}^n \partial_{x^i}^2$.

If $\boldsymbol{\alpha} \in \mathbb{Z}_+^n$ is a multi-index, we let $\boldsymbol{\nabla}^{\boldsymbol{\alpha}} = \prod_{i=1}^n \partial_{x^i}^{\alpha^i}$ and $|\boldsymbol{\alpha}| = \sum_{i=1}^n \alpha^i$.

Denoting the velocity by $\mathbf{u}(\mathbf{x}, t)$, the pressure by $p(\mathbf{x}, t)$ and the (constant) coefficient of viscosity by ν ($\nu > 0$), the Navier–Stokes equations in a domain $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, are

$$\begin{aligned} &\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \nu \Delta \mathbf{u}, \\ &\nabla \cdot \mathbf{u} = 0. \end{aligned} \tag{1.1}$$

The equations are supplemented by an initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \tag{1.2}$$

and, if $\Omega \neq \mathbb{R}^n$, by boundary conditions (such as $\mathbf{u} = 0$, the "no-slip" condition) on the boundary $\partial\Omega$, for all $t \geq 0$. If $\Omega = \mathbb{R}^n$, a growth (or, rather, decay) condition must be imposed on $\mathbf{u}$ at infinity.

These equations should yield solutions $\mathbf{u}(x, t)$, $p(\mathbf{x}, t)$, for $\mathbf{x} \in \Omega$ and all positive time $t \in \mathbb{R}_+ = (0, \infty)$. The term "well-posedness" expresses the fact that, in a suitable functional framework, the solution should not only be unique but depend continuously on the initial and boundary data.

In the case that $\mathbf{u}_0 \in L^2(\Omega)$ (or $\mathbf{u}_0 \in H^1(\Omega)$) the existence of weak solutions to the problem (strong for $H^1(\Omega)$) has been known since the pioneering work of Leray [125], (see also [128] for the case of the full plane). Strong well-posedness is only local in time if $n = 3$, and is global in time if $n = 2$. We refer to [47, 54, 121, 171] where the Galerkin approach (originally used by Leray) is expounded.

Regarding whether the system (1.1)–(1.2) is well-posed beyond the L^2 framework, we refer to [75, 116] and references therein, as well as to earlier