MLM1
Morningside Lectures in Mathematics

Lectures on the Analysis of Nonlinear Partial Differential Equations Vol. 1

非线性偏微分方程分析讲义 第一卷

Editors Fanghua Lin
Xueping Wang
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Preface

In summer of 2001, we initiated a summer school program on the harmonic analysis and its applications in nonlinear partial differential equations, with special emphases on nonlinear Schrodinger equations, kinetic equations of Bolzmann type and classical fluid equations. Over the years, there have been many distinguished mathematicians working in these fields who have come to help our program and to give series of special lectures. The program has been shown to be particularly helpful to young researchers and students. The lectures involved have gradually turned into more formal and regular seminars on *Analysis in Partial Differential Equations* at the Morningside Center of Mathematics Academy of Mathematics and System Science, Chinese Academy of Sciences.

From June 2007 to January 2008, we held a special semester PDE– program. We invited many mathematicians and experts in mathematical theory of fluid mechanics and quantum mechanics. The visitors during that period includes: Bresch Didier, Carles Rémi, Jean-Yves Chemin, Desvillettes Laurent, Lopes Filho Milton C, Nussenzveig Lopes Helena J. Novotny, Antonin, Chao-Jiang Xu, Chongchun Zeng, Ping Zhang and Yuxi Zheng, who gave a series of lectures and provided excellent lecture notes. It is no doubt that these lecture notes would be very useful for many researchers and students. In this volume, we have collected lecture notes by M. C. Lopes concerning the boundary layers of incompressible fluid flow; by C. J. Xu on the micro-local analysis and its applications to the regularities of kinetic equations; by Y. X. Zheng on the weak solutions of variational wave equation from liquid crystals, and by P. Zhang and Z. F. Zhang on the free boundary problem of Euler equations. In addition,we also included the notes by F. Nier on the hypoellipticity of Fokker-Planck operator and Witten-Laplace operator that were given earlier in the summer of 2006.

We have planned to publish in the forthcoming volumes the other lectures notes. Some are from past lectures at our program and some will be collected from the newly scheduled seminars. We hope that the publication of these lecture notes may provide valuable references and up-to-date descriptions of current developments of various related research topics, that will benefit many young researchers or graduate students. We wish to take this opportunity to thank the Morningside Center of Mathematics, the Institute of Mathematics of AMSS that

provides all necessary supports. We are particularly grateful to professor Lo Yang for his constant help, supports and encouragements to our program. We also would like to thank Guilong Gui for his careful preparations of the latex file of the entire book. We finally appreciate for the financial support from the Chinese Academy of Sciences.

Fanghua Lin in New York
Xueping Wang in Nantes
Ping Zhang in Beijing
On November 3, 2008

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Boundary Layers and the Vanishing Viscosity Limit for Incompressible 2D Flow

Milton C. Lopes Filho[1]

Abstract

This manuscript is a survey on results related to boundary layers and the vanishing viscosity limit for incompressible flow. It is the lecture notes for a 10 hour mini-course given at the Morningside Center of Mathematics, Academia Sinica, Beijing, PRC from 11/28 to 12/07, 2007. The main topics covered are: a derivation of Prandtl's boundary layer equation; an outline of the rigorous theory of Prandtl's equation, without proofs; Kato's criterion for the vanishing viscosity limit; the vanishing viscosity limit with Navier friction condition; rigorous boundary layer theory for the Navier friction condition and boundary layers for flows in a rotating cylinder.

Keywords and phrases: Incompressible flow, Navier-Stokes equations, Euler equations, vorticity, boundary layers

1. Introduction

In 1904, the issue of heavier-than-air, self propelled flight by human-made machines was at the very edge of both science and technology. The first such flight, by the Wright brothers, occurred at December 14th, 1903. A Brazilian author is honor bound to remark that a more satisfying, and better publicized "first flight" was achieved by Santos Dumont, a Brazilian living in France, in September, 1906. The flight of a fixed-wing airplane could, at least in principle, be described by near-steady, zero-viscosity, irrotational theory of airfoils, which was already available at the beginning of the twentieth century.

Classical airfoil theory explained satisfactorily the balance of forces in a wing in steady flight. In short, the force that air exerts on the wing is divided into two standard components: the lift (vertical force) and the drag (horizontal force), where horizontal means the direction of steady motion. In steady flight, these

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forces are balanced by the weight and the propulsion force. The theory predicts that the lift and the drag are proportional to the *circulation* of air velocity around the airfoil, and it was in agreement with experiments, see [1] for details. Trouble occurs when one wants to change the lift, as one should do when attempting to take off or land in a fixed wing aircraft. A theorem, due to Lord Kelvin, states that the circulation around a material curve, such as the boundary of an airfoil, is a constant of motion in ideal (i.e. non-viscous) flow — or maybe, nearly constant in slightly viscous flow. So, changing propeller speed and moving control surfaces does not change the circulation. Since airplanes start out at rest, with zero circulation around the wings, no airplane could, on theoretical ground, develop a lift, and therefore fly. Something was clearly wrong with the theory.

The correction was due to the young theoretical mechanician Ludwig Prandtl (1875 — 1953), who published a short paper in the Proceedings of the Third ICM (Heidelberg 1904) whose German title roughly means as "fluid flow in very little friction". In this article, Prandtl established a perfectly satisfactory and revolutionary explanation of the following observation:

(O) *The interaction of incompressible flow with a material boundary is completely different if the flow has very small viscosity or none at all.*

This observation, the associated explanation, called *boundary layer theory* and some of what mathematicians made of this subject in the following century and a bit, make up the subject of these lectures. For a thorough account of the development and understanding of the physics of boundary layers, we would like to refer the reader to the classical book [29]. It can be argued that this short paper by Prandtl marks the birth of modern applied mathematics.

The theory of boundary layers is a cornerstone of modern fluid mechanics, but, as in much of this field, it lacks a *rational* framework, i.e. a rigorously established connection with first principles. Although substantial mathematical work has been done in this direction, some basic questions remains unanswered. The purpose of these lectures is to probe the boundaries in the mathematical understanding of the interaction between nearly ideal flow and solid objects, perhaps to bring what is not known about this question more sharply into focus. The choice of material covered is strongly slanted towards recent work by the author and his collaborators, and it includes detailed consideration of Kato's criterion for the vanishing viscosity limit in a bounded domain, a long discussion on the vanishing viscosity limit for incompressible flows with Navier boundary condition and the detailed behavior of circularly symmetric flow inside a rotating cylinder. The choice of working with two dimensional flow is both a reasonable pedagogical choice and a comfort zone for the author — in the issue of boundary layers, the sharp distinction in behavior between 2D and 3D flows is not yet apparent, and much of the work we will discuss here generalizes readily to 3D. Finally, we mention that these notes are written thinking of a reasonably mature audience — we assume, not only familiarity with standard PDE theory, but some familiarity with the basics of mathematical fluid dynamics as well.

The remainder of these notes is divided in seven sections as follows: Section 2 contains a derivation of Prandtl's equation; Section 3 contains a broad overview of rigorous results on Prandtl's equation, including some of O. Oleinik's work, and more recent progress; Section 4 introduces and proves Kato's criterion and some related results; Section 5 is concerned with vanishing viscosity under Navier friction conditions and a proof of L^p vorticity estimates in this case; Section 6 contains an exposition on a rigorous method to treat boundary layer expansions based on ideas of geometric optics, applied to the Navier boundary condition; Section 7 explores a nearly explicit example of the behavior of the boundary layer for the no-slip condition; Section 8 contains some conclusions and open problems.

2. Prandtl's theory

In this section, we present an asymptotic derivation of Prandtl's boundary layer equation. Our point of departure is the Navier-Stokes equations, which are an expression of Newton's second law applied to the motion of a fluid, subject to an incompressibility constraint. We write

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \mu \Delta u \\ \text{div } u = 0, \end{cases} \tag{2.1}$$

where u is the fluid velocity, p is the scalar pressure and μ is the kinematic viscosity of the fluid. We have assumed that mass units have been chosen so that fluid density is one.

For this derivation, and for most of the discussion in these lectures, we will assume that we are discussing a two-dimensional fluid occupying a half plane $\mathbb{H} \equiv \{x_2 > 0\}$. Two dimensional fluids really occur either in a computer simulation, or as three dimensional fluids which are translation-invariant along some direction (often an unstable state of affairs) or as an approximate model for thin fluid layers. Much of the discussion extends very naturally to a fairly general fluid domain in three dimensions, but we will stay with the simplest possible situation for pedagogical reasons.

As we all know, problem (2.1) requires a boundary condition at $\{x_2 = 0\}$, and the condition usually deemed appropriate is the *no slip* condition $u(x_1, 0, t) = 0$. This condition expresses an assumption that viscous fluids adhere to material objects, something that is neither physically nor mathematically obvious and was subject for heated debate until the mid nineteenth century, when it became clear that it gave good agreement with experiments.

Ideal, or inviscid flow is represented by solutions of Euler's equations, which is system (2.1) with $\mu = 0$. For ideal flow, the correct boundary condition is the non-penetration condition $u_2(x_1, 0, t) = 0$. The Navier-Stokes system is a singular perturbation of the Euler system, because the small constant μ appears in front of the highest order term of (2.1). One consequence of this singular perturbation is the disparity in boundary conditions between Euler and Navier-Stokes flows - namely that u_1 at the boundary goes from being identically zero for any positive viscosity to some (in principle) nonzero function when $\mu = 0$. This disparity is the root cause of the boundary layer trouble.

Our objective here is to derive Prandtl's boundary layer equations. This is a way to quantify nearly ideal fluid behavior near the boundary by means of an appropriate set of limit equations. Let us begin with a simplifying assumption: we assume that the disparity between ideal and viscous flow is concentrated on a thin layer near $x_2 = 0$.

Our next step is to non-dimensionalize equation (2.1), using the time scale T, the length scale L for horizontal lengths, and a reference vertical length scale h. We introduce the non-dimensional constant $\nu = \mu T/L^2$, which is a measurement of the quotient of viscous by inertial forces in our flow, and measures in a physically appropriate manner how far from ideal our flow really is. The non-dimensionalization procedure simply means introducing new variables

$$\widetilde{u}^1(y,s) \equiv \frac{T}{L} u^1(Ly_1, hy_2, Ts),$$

$$\widetilde{u}^2(y,s) \equiv \frac{T}{h} u^2(Ly_1, hy_2, Ts),$$

and

$$\widetilde{p}(y,s) \equiv \frac{T^2}{L^2} p(Ly_1, hy_2, Ts).$$

which results in the system

$$\begin{cases} \partial_s \widetilde{u}^1 + \widetilde{u} \cdot \nabla_y \widetilde{u}^1 = -\partial_{y_1} \widetilde{p} + \nu \left[\partial^2_{y_1} \widetilde{u}^1 + \dfrac{L^2}{h^2} \partial^2_{y_2} \widetilde{u}^1 \right] \\ \partial_s \widetilde{u}^2 + \widetilde{u} \cdot \nabla_y \widetilde{u}^2 = -\dfrac{L^2}{h^2} \partial_{y_2} \widetilde{p} + \nu \left[\partial^2_{y_1} \widetilde{u}^2 + \dfrac{L^2}{h} \partial^2_{y_2} \widetilde{u}^2 \right] \\ \mathrm{div}_y \widetilde{u} = 0. \end{cases} \tag{2.2}$$

We introduce $\varepsilon \equiv h/L$, which is assumed to be a small, non-dimensional, parameter because we are focusing in a thin layer. We also want to consider ν small. A key issue in boundary layer theory is that the magnitude of the small parameters ε and ν are naturally related. Indeed, if $\nu \ll \varepsilon^2$ then matched asymptotics indicates that, to leading order, $\widetilde{u}$ satisfies the Euler system (with pressure independent of the vertical variable) and with no-slip conditions at the boundary. These boundary conditions are inconsistent with the fact that the Euler system is of first order. On the other hand, if $\nu \gg \varepsilon^2$ then, to leading order, $\widetilde{u}$ is such that $\partial^2_{y_2} \widetilde{u} = 0$, which, together with the no-slip boundary condition implies that $\widetilde{u} = c(y_1) y_2$, for some vector c. Now, if $\widetilde{u}$ is to represent the behavior of the flow in a thin layer near the boundary, then the velocity $\widetilde{u}$ should match the inviscid velocity in the limit $y_2 \to \infty$, and not just blow-up. The only regime that appears to yield a consistent asymptotic regime is

$$\nu/\varepsilon^2 = \mathcal{O}(1). \tag{2.3}$$

From another perspective, condition (2.3) highlights the region near the boundary where the vertical viscous stress balances the inertial terms in the Navier-Stokes system. Assuming $\nu = \varepsilon^2$ and implementing matching asymptotics for ν small we obtain the following system for the leading order approximation, denoted $v = (v^1, v^2)$,

$$\begin{cases} \partial_s v^1 + v \cdot \nabla_y v^1 = -\partial_{y_1} q + \partial_{y_2}^2 v^1 \\ \partial_{y_2} q = 0 \\ \mathrm{div}_y v = 0. \end{cases} \tag{2.4}$$

These are the *unsteady Prandtl equations* for the boundary layer profile v. They represent the behavior of the flow near the boundary. To obtain a complete problem, these equations must be supplemented with boundary conditions. First we impose the no-slip condition

$$v = 0 \text{ at } y_2 = 0.$$

An additional condition must be imposed in order to capture the assumption that far from the boundary layer the small viscosity Navier-Stokes solutions match the Euler solutions. Let u^ν a family of solutions of the non-dimensional Navier-Stokes equations with non-dimensional viscosity ν and u^E be a solution of the incompressible Euler equations. For example, we can assume that both the family u^ν and u^E are defined by solving the Navier-Stokes equations and the Euler equations with the same initial data $u^\nu(x,0) = u^E(x,0) = u_0(x)$.

Going back to the Prandtl system, we expect that $v(y_1, y_2, t) \to u^E(y_1, 0, t) \equiv U(y_1, t)$, as $y_2 \to \infty$. Let p^E be the pressure associated with the Euler solution u^E. Since q does not depend on y_2, looking at $y_2 \to \infty$ makes it also natural to assume that $q(y_1, t) = p^E(y_1, 0, t)$. If we look at the Euler equations and evaluate them at $x_2 = 0$ we obtain the following relation:

$$U_t + UU_{y_1} = -P^E_{y_1} = -q_{y_1},$$

which is called Bernoulli's Law. This means that, for the Prandtl equation (2.4), the condition at infinity U determines, up to an irrelevant constant, the pressure q.

With this construction, we hope that, when ν is small,

$$u^\nu(x_1, x_2, t) = \begin{cases} (v^1, \varepsilon v^2)(x_1, x_2/\varepsilon, t) \text{ for } x_2 < \lambda(\nu) \\ u^E(x_1, x_2, t) \text{ for } x_2 > \lambda(\nu) \end{cases} + o(1), \tag{2.5}$$

where $\lambda(\nu)$ is any cutoff distance such that $\varepsilon \ll \lambda(\nu) \ll 1$.

In addition to the time-dependent Prandtl equation, this derivation also yields the *steady Prandtl equation*, given by

$$\begin{cases} v \cdot \nabla_y v^1 = -\partial_{y_1} q + \partial_{y_2}^2 v^1 \\ \partial_{y_2} q = 0 \\ \mathrm{div}_y v = 0. \end{cases} \tag{2.6}$$

The typical problem associated with this equation is a quarter-plane BVP, where a profile $v(0, y_2)$ is given and one attempts to find $v(y_1, y_2)$ for $y_1 > 0$ as the induced boundary layer profile over a half-plane plate.

The derivation above is a nice example of multiscale asymptotic analysis, and from the complicated issue surrounding the interaction of nearly inviscid flow with

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material boundaries it derives a new equation, (2.4), and a simplified asymptotic model for the behavior of Navier-Stokes solutions near a material boundary, given by (2.5). The key issue that such a model raises is its validity (mathematical) and applicability (physical).

This model has been found useful in applications, specially where it concerns laminar boundary layers, and when its usefulness begins to break down, suitable extensions of the model have been obtained, notably the so called "triple deck" expansions, where an intermediate thin layer is added between the viscosity-dominated internal layer and the free irrotational Euler flow. In this intermediate layer, the flow is ideal, but not necessarily irrotational. One important situation where the boundary layer ansatz breaks down is when *boundary layer separation* occurs. Recall that one of the assumptions in deriving the Prandtl equation was that the disparity between ideal and viscous flow be concentrates in a thin layer *near the boundary.* It is quite common, even well within laminar flow regimes, that the boundary layer detaches itself from the boundary and affects the bulk of the flow. In that case, Prandtl's theory and its extensions break down as models. In the next section we will consider what is known regarding the rigorous validation of the asymptotic approximation described here.

3. Prandtl's equation

The purpose of this section is to present the known theory for Prandtl's equation, without much detail. The first question one must address regarding an approximate model is: can I solve it? The initial and boundary value problems for the Euler and Navier-Stokes equations are well-posed, globally in the case of the half-plane with reasonable initial data. So the issue of whether Euler + Prandtl is a good approximation for Navier-Stokes with ν small, in the sense discussed in the previous section, depends first on understanding the well-posedness for Prandtl's equation. The mathematical theory of the Prandtl equation only got started in the sixties, by O. Oleinik. Over the years, Oleinik and her group made a large number of contributions to the theory of Prandtl's equation and its many variants, collected and explained in the book [27]. For the present discussion, we would like to focus on one specific result, that first appeared in [26]. We also refer the reader to the survey [5], for the discussion of Oleinik's result and its relation with the blow-up result of W. E and B. Engquist, [6].

Let $v = (v^1, v^2)$ be a solution of the IBVP for Prandtl's equation, which we write as

$$\begin{cases} \partial_t v^1 + v \cdot \nabla v^1 = -\partial_{x_1} q + \partial_{x_2}^2 v^1 \\ \operatorname{div} v = 0 \\ v(x_1, 0, t) = 0 \text{ and } \lim_{x_2 \to \infty} v^1(x,t) = U(x_1, t) \\ v(x, 0) = v_0(x), \end{cases} \tag{3.1}$$

Where $-\partial_{x_1} q = U_t + U U_{x_1}$. The result we wish to discuss is the following

Theorem 3.1. (Oleinik 1967) *Assume that both U and v_0^1 are positive and that, in addition, $\partial_{x_2} v_0^1 \geq 0$. Then there exists a unique global strong solution of* (3.1).

We will not present a proof of this result, but we will discuss a key part of the proof, which is the recasting of this problem as a scalar, degenerate parabolic scalar equation, using Crocco's transformation.

We begin by taking the derivative of Prandtl's equation with respect to x_2 and introduce the new dependent variable $\omega(x,t) \equiv \partial_{x_2} v^1$. System (3.1) is equivalent to the following IBVP:

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = \partial_{x_2}^2 \omega \\ v = K[\omega] \\ (\partial_{x_2}\omega)(x_1, 0, t) = \partial_{x_1} q \text{ and } \lim_{x_2 \to \infty} \omega(x,t) = 0 \\ \omega(x, 0) = \partial_{x_2} v_0(x), \end{cases} \tag{3.2}$$

where the vector operator K reconstructs (v^1, v^2) by first integrating in the vertical variable to obtain v_1 and then using the divergence-free condition and integrating again in the vertical variable to obtain v_2. The new equation (3.2) is, in a sense, the vorticity formulation of problem (3.1).

We assume that the solution $v = v(x,t)$ we seek satisfies the condition $\partial_{x_2} v^1(x,t) > 0$ for all $x \in \mathbb{H}, t > 0$. In particular, this means that, for each fixed (x_1, t), the map $x_2 \mapsto v^1(x_1, x_2, t)$ is invertible. Let us denote this inverse by $h = h(x_1, \xi, t)$. In other words, we have

$$v^1(x_1, h(x_1, \xi, t), t) = \xi, \text{ for all } \xi > 0. \tag{3.3}$$

The Crocco's transform consists of introducing the new dependent variable:

$$W = W(x_1, \xi, t) \equiv \omega(x_1, h(x_1, \xi, t), t). \tag{3.4}$$

We verify that W is a solution of the following IBVP:

$$\begin{cases} \partial_t W + \xi W_{x_1} - (\partial_{x_1} q) W_\xi = W^2 \partial_\xi^2 W \\ W W_\xi = q_{x_1} \text{ for } \xi = 0 \\ W(x, 0) = (\partial_{x_2} v_0)(x_1, h(x_1, \xi, 0)). \end{cases} \tag{3.5}$$

Indeed, we can compute directly to obtain:

$$\partial_t W = -\frac{v_t^1}{\omega}\omega_{x_2} + \omega_t; \; \partial_{x_1} W = -\frac{v_{x_1}^1}{\omega}\omega_{x_2} + \omega_{x_1}; \; \partial_\xi W = \frac{\omega_{x_2}}{\omega}; \; \partial_\xi^2 W = \frac{\omega_{x_2 x_2}}{\omega^2} - \frac{\omega_{x_2}^2}{\omega^3}.$$

Substituting the corresponding equalities above into (3.5), and using (3.4), (3.3) and the evolution equations in (3.1) and (3.2) verifies the evolution equation in (3.5). In addition, we can check directly that

$$\omega_{x_2}(x_1, x_2, t) = W(x_1, g^{-1}(x_1, x_2, t), t) W_\xi(x_1, g^{-1}(x_1, x_2, t), t),$$

which, together with the boundary condition in (3.2) gives the boundary condition in (3.5). Problem (3.5) is a scalar, degenerate parabolic equation, which is amenable to fairly standard treatment, using fixed point methods, and satisfies a comparison principle. In particular, the sign of W is retained in the evolution, and therefore, the monotonicity condition on v^1, necessary for the validity of Crocco's transform, is retained as well.

Finally, once a solution W is obtained for problem (3.5), one must reconstruct a solution to the original problem. Recall that

$$v^1(x, g(x_1, \xi, t), t) = \xi,$$

and therefore, differentiating this identity with respect to ξ gives

$$\omega(x_1, g(x_1, \xi, t), t)\frac{\partial g}{\partial \xi}(x_1, \xi, t) = 1.$$

Recalling (3.4), this implies that

$$g(x_1, \xi, t) = \int_0^\xi (W(x_1, \eta, t))^{-1} d\eta,$$

which allows the reconstruction of ω from W by means of (3.4). The interested reader may prove, as an exercise, that such an ω is, in fact, a solution of (3.2).

This result, and others proved by Oleinik and her group, give useful, rigorously established descriptions of the vanishing viscosity asymptotics, but depend, to a greater or lesser extent, on monotonicity conditions such as $\partial_{x_2} v_0^1 \geq 0$. As we have seen, the monotonicity assumption is needed for the validity of the Crocco's transformation, but this assumption might just be a feature of the method, rather than an essential limitation of the theory. In 1997, E and Engquist produced a counterexample which showed that Prandtl's equation develops finite-time singularities if the monotonicity condition is not imposed, see [6]. In fact, E and Engquist's example suggests that the role of the monotonicity assumption is to prevent boundary layer separation, a phenomenon that actually occurs in real flows and corresponds to a breakdown of the Prandtl ansatz.

An alternative to the half-plane analysis described above is to study well-posedness of Prandtl's equation in bounded intervals in x_1, where the horizontal velocity of the boundary layer profile is specified in one side of the interval, and the length of the interval or the time horizon of the analysis are chosen small enough to prevent separation. Such a result was first proved by Oleinik in [25]. Recently, Z. Xin and L. Zhang improved Oleinik's result, showing existence of a global (in time) weak solution for Prandtl's equation on a finite horizontal interval, if the pressure is *favorable*, i.e., $q_{x_1} < 0$, see [34] and [35]. This condition is also known to discourage boundary layer separation.

A different approach to the theory of Prandtl's equation was taken, initially by A. Asano, in a couple of unpublished manuscripts, and later by R. Caflisch and M. Sammartino, in a pair of articles, see [28], recently further improved by Lombardo, Cannone and Sammartino in [17]. The basic idea is that, without the monotonicity condition, or something analogous to it, one expects the initial-boundary value problem (3.1) to be ill-posed. As a result, it becomes natural to look for local (in time) solutions for Prandtl's equation in analytic function spaces, using results of Cauchy-Kowalewska type. The main results in [28] were well-posedness of the problem (3.1) if the data v_0 and U are analytic, and compatible. In [17] the analyticity requirement on v_0 was imposed only in the horizontal variables.

Of course, the well-posedness in analytic spaces, and the blow-up example by E and Engquist does not prove that (3.1) is ill-posed, which at this time remains an interesting open problem.

To conclude this section, it would make sense to mention the contribution of E. Grenier, which he describes as nonlinear instability of the Prandtl boundary layer. His result is not about Prandtl's equation per se, but about the vanishing viscosity limit of the Navier-Stokes equations. His result can be interpreted as mathematical evidence that the Prandtl ansatz is not always valid for solutions of the Navier-Stokes system in the half-plane with small viscosity, see [7]. In other words, although the theory of Prandtl's equation is relevant for understanding the vanishing viscosity limit, there is more to the original observation (O) than Prandtl's original explanation for it.

4. Kato's condition

In this section we move away from Prandtl's equation, and we begin a study of the vanishing viscosity limit from a broader point of view. Our first observation should be that, even in the absence of boundaries, all the mysteries of turbulence lurk in the background of the vanishing viscosity limit, see for example [18] for a small part of this story. However, under moderate regularity assumptions, for example, if the initial vorticity is bounded, explicit estimates for the difference between Euler and small viscosity Navier-Stokes solutions are known, see [2]. Also, and this distinction is a key point here, for very irregular flow, we still have the existence of subsequences of solutions of small viscosity Navier-Stokes converging to weak solutions of the Euler equations, up to and including initial vorticities which are measures, see [23, 22], but then no estimates on the difference are provided, or expected. Basically, in the absence of boundaries, as long as the underlying ideal flow has enough regularity so that uniqueness of weak solutions to the Euler equations is known, we have actual convergence of the vanishing viscosity limit. Furthermore, as long as existence of weak solutions is known we also have compactness of the vanishing viscosity sequence and weak continuity of the Euler/Navier-Stokes nonlinearity. Nonuniqueness of weak solutions for Euler equations is also known, see the remarkable paper [4], and references therein, for the current knowledge on this nonuniqueness, but the behavior of the vanishing viscosity limit for these examples is a very interesting open problem.

As soon as we consider flows in the presence of boundaries, the story changes quite dramatically. Very little is actually known mathematically, and this very little is precisely the object under discussion in these notes. Physically, boundaries are the most natural source of small scales in incompressible flows, precisely through the boundary layer mechanism, and these small scales are the source of the irregularities that justify considering irregular 2D flows in the first place. The point of departure in our discussion will be a classical open problem, which we formulate below.

∂ Layer Problem: Let u^ν be a sequence of solutions of the incompressible

Navier-Stokes equations in two space dimensions, in a smooth bounded domain Ω, satisfying the no-slip boundary condition on $\partial\Omega$, with initial data u_0^ν, bounded in L^2. Is there a subsequence u^{ν_k} converging weakly in L^2 to a vector field u, which is a weak solution of the incompressible Euler equations in Ω with some initial data $u_0 = \lim u_0^{\nu_k}$?

This problem is open even if $\omega_0^\nu = \omega_0 \in C_c^\infty(\Omega)$, with $\omega_0 =$ curl u_0. Let us focus, for simplicity, in this case. Clearly, the Navier-Stokes equations have a unique smooth solution u^ν with initial data u_0, and the Euler equations also have a unique smooth solution u with the same initial data. We will see that there are examples where $u^\nu \to u$ in L^2, but the answer to the problem above may be positive even when u^ν does not converge to u, because there may be weak solutions of the incompressible Euler equations with initial velocity u_0 which are not u.

In 1984, T. Kato wrote a short note where he proved a sharp criterion for the convergence of u^ν to u, see [11]. The observation by Kato is remarkable for at least two reasons. First, as we shall see, it is very natural from the analytical point of view. Second, it places the condition for convergence on the behavior of the small viscosity sequence at a distance $\mathcal{O}(\nu)$ of the boundary of Ω, hence in a much smaller region than what is the natural domain of the boundary layer. Next, we state and prove a simple version of Kato's criterion.

We focus on the case $\Omega = \{|x| < 1\}$ in $\mathbb{R}^2$. Let $\omega_0 \in C_c^\infty(\Omega)$ and $u_0 \equiv K[\omega_0]$, where K is the Biot-Savart operator in the unit disk. Let u^ν be the unique classical solution of the Navier-Stokes equation in Ω with no-slip boundary condition and initial velocity u_0, and u be the unique smooth solution of the Euler equations with $u \cdot x = 0$ for $|x| = 1$ and initial velocity u_0.

Theorem 4.1. (Kato 1984) *Fix $T > 0$. There exists a constant $c > 0$ such that $u^\nu \to u$ strongly in $L^\infty((0,T); L^2(\Omega))$ if and only if $\nu \int_0^T \|\nabla u^\nu(\cdot,t)\|^2_{L^2(\Gamma_{c\nu})} dt \to 0$ as $\nu \to 0$, where $\Gamma_{c\nu} \equiv \{1 - c\nu < |x| < 1\}$.*

Proof: First consider the energy identities for both u^ν and u. We have, for each $t > 0$,

$$\|u^\nu(\cdot,t)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)} + \nu \int_0^t \int_\Omega |\nabla u^\nu|^2 dx dt,$$

and

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}.$$

Therefore, if $u^\nu \to u$ strongly in $L^\infty((0,T); L^2(\Omega))$, then $\|u^\nu(\cdot,t)\|^2_{L^2(\Omega)} \to \|u(\cdot,t)\|^2_{L^2(\Omega)}$ for almost all time, and therefore

$$\nu \int_0^t \int_\Omega |\nabla u^\nu|^2 dx dt \to 0,$$

for each $t > 0$, not almost everywhere anymore since the integral in time is increasing in time, and therefore,

$$\nu \int_0^t \int_{\Gamma_{c\nu}} |\nabla u^\nu|^2 dx dt \to 0,$$

as we wished.

To prove the other implication, fix $\varepsilon > 0$ and let $\phi^\varepsilon \in C_c^\infty(\Omega)$ be such that $\phi^\varepsilon(x) = \varphi^\varepsilon(|x|)$, with $\varphi^\varepsilon(x) = 1$ for $|x| < 1 - \varepsilon$, $\varphi^\varepsilon(x) = 0$ for $1 - \varepsilon/2 < |x| \leq 1$, and φ^ε decreases monotonically from 1 to 0. Let $\omega =$ curl u be the vorticity and ψ be the stream function associated with the Euler flow u. Define

$$u_\varepsilon = \nabla^\perp(\phi^\varepsilon\psi) = (-\partial_{x_2}(\phi^\varepsilon\psi), \partial_{x_1}(\phi^\varepsilon\psi)).$$

Let $v_\varepsilon \equiv u - u_\varepsilon = \nabla^\perp((1-\phi^\varepsilon)\psi)$. The stream function ψ vanishes at $|x| = 1$, and it can be assumed to be uniformly bounded in C^k, for any $k = 1, 2, \ldots$, so that we can easily obtain the following estimates on v_ε:

$$\|v_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C\varepsilon^{1/2} \tag{4.1}$$

$$\|\partial_t v_\varepsilon\|_{L^1((0,T);L^2(\Omega))} \leq C\varepsilon^{1/2} \tag{4.2}$$

$$\|\nabla v_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} \leq C\varepsilon^{-1/2} \tag{4.3}$$

$$\|v_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq C \tag{4.4}$$

$$\|\nabla v_\varepsilon\|_{L^\infty((0,T)\times\Omega)} \leq C\varepsilon^{-1} \tag{4.5}$$

In addition, we require certain estimates on u^ν, uniform in ν, which we collect below

$$\|u^\nu\|_{L^\infty((0,T);L^2(\Omega))} \leq C \tag{4.6}$$

$$\nu\|\nabla u^\nu\|^2_{L^\infty((0,T);L^2(\Omega))} \leq C. \tag{4.7}$$

By using Cauchy-Schwarz in time, we also have

$$\nu^{1/2}\|\nabla u^\nu\|_{L^1((0,T);L^2(\Omega))} \leq CT^{1/2}\left(\nu\int_0^T \|\nabla u^\nu\|^2_{L^2(\Omega)}dt\right)^{1/2} \leq C \tag{4.8}$$

Finally, a version of Poincaré's Inequality, which reads

$$\|u^\nu\|_{L^2(\Gamma_\varepsilon)} \leq C\varepsilon\|\nabla u^\nu\|_{L^2(\Gamma_\varepsilon)} \tag{4.9}$$

Now we estimate, omitting the explicit dependence of v on ε:

$$\|u^\nu - u\|^2_{L^2(\Omega)} = \|u^\nu\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} - 2\langle u^\nu, u\rangle$$

$$\leq 2\|u_0\|^2_{L^2(\Omega)} - 2\langle u^\nu, u - v\rangle - 2\langle u^\nu, v\rangle.$$

We have that

$$|\langle u^\nu, v\rangle| \leq \|u^\nu\|_{L^\infty((0,T);L^2(\Omega))}\|v\|_{L^\infty((0,T);L^2(\Omega))} \leq C\varepsilon^{1/2}.$$

And therefore, taking $\varepsilon = C\nu$,

$$\|u^\nu - u\|^2_{L^2(\Omega)} \leq 2\|u_0\|^2_{L^2(\Omega)} - 2\langle u^\nu, u - v\rangle + o(1). \tag{4.10}$$

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