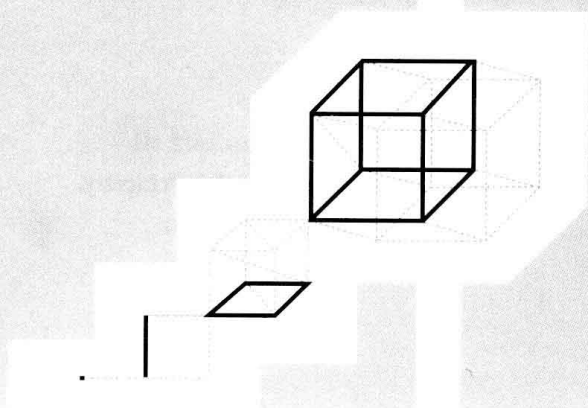


I-Hsiung Lin

GEOMETRIC

LINEAR **A**LGEBRA

Volume 2



GEOMETRIC LINEAR ALGEBRA

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I-Hsiung Lin

National Taiwan Normal University, China

 **World Scientific**

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.

5 Toh Tuck Link, Singapore 596224

USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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(Volume 2)

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ISBN-13 978-981-270-775-8 (pbk)

ISBN-10 981-270-775-1 (pbk)

Printed in Singapore by World Scientific Printers

GEOMETRIC LINEAR ALGEBRA

Volume **2**

To the memory of my
grandparents, parents
and to my family

PREFACE TO VOLUME ONE

What is linear algebra about?

Many objects such as buildings, furniture etc. in our physical world are delicately constituted by counterparts of almost straight and flat shapes which, in geometrical terminology, are portions of straight lines or planes. A butcher's customers ordered the meat in varying quantities, some by mass and some by price, so that he had to find answers to such questions as: What is the cost of $2/3$ kg of the meat? What mass a customer should have if she only wants to spend 10 dollars? This can be solved by the linear relation $y = ax$ or $y = ax + b$ with b as a bonus. The same is, when traveling abroad, to know the value of a foreign currency in term of one's own. How many faces does a polyhedron with 30 vertices and 50 edges have? What is the Fahrenheit equivalent of 25°C ? One experiences numerous phenomena in daily life, which can be put in the realms of straight lines or planes or linear relations of several unknowns.

The most fundamental and essential ideas needed in *geometry* are

1. (directed) line segment (including the extended line),
2. parallelogram (including the extended plane)

and the associated quantities such as length or signed length of line segment and angle between segments.

The *algebraic* equivalence, in global sense, are linear equations such as

$$a_{11}x_1 + a_{21}x_2 = b_1$$

or

$$a_{11}x_1 + a_{21}x_2 + a_{31}x_3 = b_1$$

etc. and simultaneous equations composed of them. The core is how to determine whether such linear equations have a solution or solutions, and if so, how to find them in an effective way.

Algebra has operational priority over geometry, while the latter provides intuitively geometric interpretations to results of the former. Both play a role of head and tail of a coin in many situations.

Linear algebra is going to transform the afore-mentioned geometric ideas into two algebraic *operations* so that solving linear equations can be handled *linearly* and systematically. Its implication is far-reaching and it's application is widely open and touches almost every field in modern science. More precisely,

a directed line segment $\overrightarrow{AB} \rightarrow$ a vector \vec{x} ;

ratio of signed lengths of (directed) line segments along the same line

$$\frac{\overrightarrow{PQ}}{\overrightarrow{AB}} = \alpha \rightarrow$$

$\vec{y} = \alpha \vec{x}$, *scalar multiplication* of \vec{x} by α .

See Fig. P-1.

Hence, the whole line can be described algebraically as $\alpha \vec{x}$ while α runs through the real numbers. While, the parallelogram in Fig. P-2 indicates that directed segments \overrightarrow{OA} and \overrightarrow{BC} represent the same vector \vec{x} , \overrightarrow{OB} , and \overrightarrow{AC} represent the same vector \vec{y} so that

the diagonal $\overrightarrow{OC} \rightarrow \vec{x} + \vec{y}$, the *addition* of vectors \vec{x} and \vec{y} .

As a consequence, the whole plane can be described algebraically as the linear combinations $\alpha \vec{x} + \beta \vec{y}$ where α and β are taken from all the reals. In fact, parallelograms provide implicitly as an inductive process to construct and visualize higher dimensional spaces. One may imagine the line OA

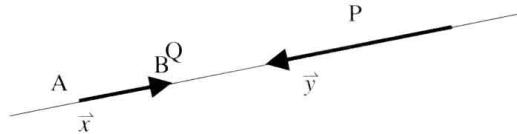


Fig. P-1

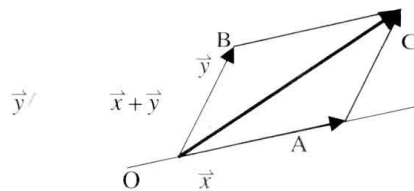


Fig. P-2

acting as an $(n-1)$ -dimensional space, so that \vec{x} is of the form $\alpha_1 \vec{x}_1 + \cdots + \alpha_{n-1} \vec{x}_{n-1}$. In case the point C is outside the space, \vec{y} cannot be expressed as such linear combinations. Then the addition $\alpha \vec{y} + \vec{x}$ will raise the space to the higher n -dimensional one.

As a whole, relying only on

$\alpha \vec{x}$, $\alpha \in \mathbb{R}$ (real field) and

$\vec{x} + \vec{y}$

with appropriate operational properties and using the techniques:

linear combination,

linear dependence, and

linear independence

of vectors, plus deductive and inductive methods, one can develop and establish the whole theory of Linear Algebra, even formally and in a very abstract manner.

The main theme of the theory is about linear transformation which can be characterized as the mapping that *preserves* the ratio of the signed lengths of directed line segments along the same or parallel lines. Linear transformations between finite-dimensional vector spaces can be expressed as matrix equations $\vec{x} A = \vec{y}$, after choosing suitable coordinate systems as bases.

The matrix equation $\vec{x} A = \vec{y}$ has two main features. The *static structure* of it, when consider \vec{y} as a constant vector \vec{b} , results from solving algebraically the system $\vec{x} A = \vec{b}$ of linear equations by the powerful and useful Gaussian elimination method. Rank of a matrix and its factorization as a product of simpler ones are the most important results among all. Rank provides insights into the geometric character of subspaces based on the concepts of linear combination, dependence, and independence. While factorization makes the introduction of determinant easier and provides preparatory tools to understand another feature of matrices. The *dynamic structure*, when consider \vec{y} as a varying vector, results from treating A as a linear transformation defined by $\vec{x} \rightarrow \vec{x} A = \vec{y}$. The kernel and range of a linear transformation, dimension theorem, invariant subspaces, diagonalizability, various decompositions of spaces or linear transformations and their canonical forms are the main topics among others.

When Euclidean concepts such as lengths and angles come into play, it is the inner product that combines both and the Pythagorean Theorem or orthogonality dominates everywhere. Therefore, linear operators $\vec{y} = \vec{x} A$

are much more specified and results concerned more fruitful, and provide wide and concrete applications in many fields.

Roughly speaking, using algebraic methods, linear algebra investigates the possibility and how of solving system of linear equations, or geometrically equivalent, studies the inner structures of spaces such as lines or planes and possible interactions between them.

The Purpose of This Introductory Book

The teaching of linear algebra and its contents has become too algebraic and hence too abstract, at least, in the introduction of main concepts and the methods which are going to become formal and well established in the theory. Too fast abstraction of the theory definitely scares away many students whose majors are not in mathematics but need linear algebra very much in their careers.

For most beginners in a first course of linear algebra, the understanding of clearer pictures or the reasons why to do this and that seems more urgent and persuasive than the rigorousness of proofs and the completeness of the theory. Understanding cultivates interestingness to the subject and abilities of computation and abstraction.

To start from one's knowledge and experience does enhance the understanding of a new subject. As far as beginning linear algebra is concerned, I strongly believe that intuitive, even manipulatable, geometric objects or concepts are the best ways to open the gate of entrance. This is the momentum and the purpose behind the writing of this introductory book. I tried before (in Chinese) and I am trying to write this book in this manner, maybe not so successful as originally expected but away from the conventional style in quite a few places (refer to *Appendix B*).

This book is designed for beginners, like freshman and sophomore or honored high school students.

The general prerequisites to read this book are high-school algebra and geometry. Appendix A, which discuss sets, functions, fields, groups and polynomials, respectively, are intended to unify and review some basic ideas used throughout the book.

Features of the Book

Most parts of the contents of this book are abridged briefly from my seven books on *The Introduction to Elementary Linear Algebra* (published in

Chinese from 1982 to 1984, with the last two still unable to be published until now). I try to write the book in the following manner:

1. Use intuitive geometric concepts or methods to introduce or to motivate or to reinforce the creation of abstract or general theory in linear algebra.
2. Emphasize the geometric characterizations of results in linear algebra.
3. Apply known results in linear algebra to describe various geometries based on F. Klein's Erlanger' point of view.

Therefore, in order to vivify these connections of geometries with linear algebra in a convincing argument, I focus the discussion of the whole book on the real vector space \mathbb{R}^1 , \mathbb{R}^2 , and \mathbb{R}^3 endowed with more than 500 *graphic illustrations*. It is in this sense that I label this book the title as *Geometric Linear Algebra*. Almost each section is followed by a set of exercises.

4. Usually, each **Exercises** contains two parts: $\langle A \rangle$ and $\langle B \rangle$. The former is designed to familiarize the readers with or to practice the established results in that section, while the latter contains challenging ones whose solutions, in many cases, need some knowledge to be exposed formally in sections that follow. In addition to these, some **Exercises** also contain parts $\langle C \rangle$ and $\langle D \rangle$. $\langle C \rangle$ asks the readers to try to model after the content and to extend the process and results to vector spaces over arbitrary fields. $\langle D \rangle$ presents problems concerned with linear algebra, such as in real or complex calculus, differential equations and differential geometry, etc.

The readers are asked to do all problems in $\langle A \rangle$ and are encouraged to try part in $\langle B \rangle$, while $\langle C \rangle$ and $\langle D \rangle$ are optional and are left to more mature and serious students.

No applications outside pure mathematics are touched and the needed readers should consult books such as Gilbert Strang's *linear Algebra and its Application*.

Finally, three points deviated from most existed conventional books on linear algebra should be cautioned. One is that chapters are divided according to affine, linear, and Euclidian structures of \mathbb{R}^1 , \mathbb{R}^2 and \mathbb{R}^3 , but not according to topics such as vectors spaces, determinants, etc. The other is that few definitions are formal and most of them are allowed to come to the surface in the middle of discussions, while main results obtained after a discussion are summarized and typed in bold-face and are numbered along with important formulas. The third one is that a point $\vec{x} = (x_1, x_2)$ is

also treated as a position *vector* from the origin $\vec{0} = (0, 0)$ to that point, when \mathbb{R}^2 is considered as a two-dimensional vector space, rather than the common used notation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ or } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

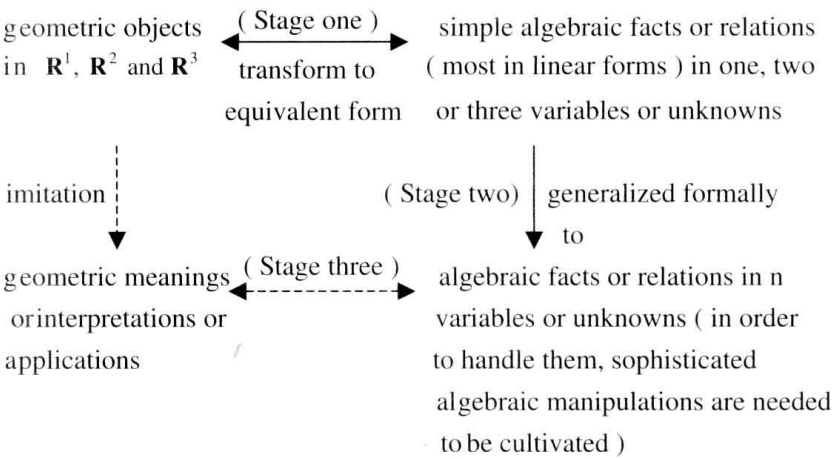
As a consequence of this convention, when a given 2×2 matrix A is considered to represent a linear transformation on \mathbb{R}^2 to act on the vector \vec{x} , we adopt $\vec{x} A$ but not $A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ to denote the image vector of \vec{x} under A . Similar explanation is valid for $\vec{x} A$ where $\vec{x} \in \mathbb{R}^m$ and A is an $m \times n$ matrix, etc.

In order to avoid getting lost and for a striking contrast, I compensate *Appendix B* and title it as *Fundamentals of Algebraic Linear Algebra* for the sake of reference and comparison.

Ways of writing and how to treat \mathbb{R}^n for $n \geq 4$

The main contents are focused on the introduction of \mathbb{R}^1 , \mathbb{R}^2 , and \mathbb{R}^3 , even though the results so obtained and the methods cultivated can almost be generalized verbatim to \mathbb{R}^n for $n \geq 4$ or finite-dimension vector spaces over fields and, in many occasions, even to infinite-dimensional spaces.

As mentioned earlier, geometric motivation will lead the way of introduction to the well-established and formulated methods in the contents. So the general process of writing is as follows:



In most cases, we leave Stages two and three as Exercises ⟨C⟩ for mature students.

As a whole, we can use the following *problem*:

Prove the identity $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$, where n is any natural number, by the mathematical induction.

as a model to describe how we probe deeply into the construction and formulation of topics in *abstract* linear algebra. We proceed as follows. It is a dull business at the very beginning to prove this identity by testing both sides in cases $n = 1, n = 2, \dots$ and then supposing both sides equal to each other in case $n = k$ and finally trying to show both sides equal when $n = k + 1$. This is a well-established and sophisticated way of arguments, but it is not necessarily the best way to understand thoroughly the implications and the educational values this problem could provide. Instead, why not try the following steps:

1. How does one know beforehand that the sum of the left side is equal to $\frac{1}{6}n(n+1)(2n+1)$?
2. To pursue this answer, try trivial yet simpler cases when $n = 1, 2, 3$ and even $n = 4$, and then try to find out possible common rules owned by all of them.
3. Conjecture that the common rules found are still valid for general n .
4. Try to prove this conjecture formally by mathematical induction or some other methods.

Now, for $n = 1$, take a “shadow” unit square and a “white” unit square and put them side by side as Fig. P-3:

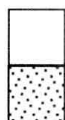


Fig. P-3

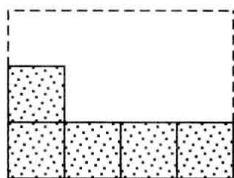


Fig. P-4

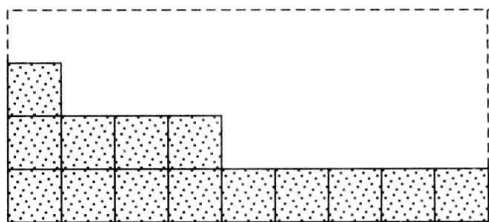


Fig. P-5

This suggests the conjecture

$$\frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{(n+1)n^2} = \frac{2n+1}{6n}$$

$$\Rightarrow 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1).$$

It is approximately in this manner that I wrote the contents of the book, in particular, in Chaps. 1, 2 and 4. Of course, this procedure is roundabout, overlapped badly in some cases and even makes one feel impatiently and sick. So I tried to summarize key points and main results on time. But I do strongly believe that it is a worth way of educating beginners in a course of linear algebra.

Well, I am not able to realize physically the existence of four or higher dimensional spaces. Could you? How? It is algebraic method that convinces us properly the existence of higher dimensional spaces. Let me end up this puzzle with my own experience in the following story.

Some day in 1986, in a Taoism Temple in eastern Taiwan, I had a face-to-face dialogue with a person epiphanized (namely, making its presence or power felt) by the god Nuo Zha (also called the Third Prince in Chinese communities):

I asked: Does there exist god?

Nuo Zha answered: Gods do exist and they live in spaces, from dimension seven to dimension thirteen. You common human being lives in dimension three, while dimensions four, five, and six are buffer zones between human being and gods. Also, there are "human being" in underearth, which are two-dimensional.

I asked: Does UFO (unfamiliar objects) really exist?

Nuo Zha answered: Yes. They steer improperly and fall into the three-dimensional space so that you human being can see them physically. Believe it or not!

Sketch of the contents

Catch a quick glimpse of the *Contents* or *sketch of the content* at the beginning of each chapter and one will have rough ideas about what might be going on inside the book.

Let us start from an example.

Fix a Cartesian coordinate system in space. Equation of a plane in space is $a_1x_1 + a_2x_2 + a_3x_3 = b$ with $b = 0$ if and only if the plane passes through the origin $(0,0,0)$. Geometrically, the planes $a_1x_1 + a_2x_2 + a_3x_3 = b$ and $a_1x_1 + a_2x_2 + a_3x_3 = 0$ are parallel to each other and they will be coincident by invoking a translation. Algebraically, the main advantage of a plane passing through the origin over these that are not is that it can be vectorized as a two-dimensional vector space $\langle\langle \vec{v}_1, \vec{v}_2 \rangle\rangle = \{\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 | \alpha_1, \alpha_2 \in \mathcal{R}\}$, where \vec{v}_1 and \vec{v}_2 are linear independent *vectors* lying on $a_1x_1 + a_2x_2 + a_3x_3 = 0$, while $a_1x_1 + a_2x_2 + a_3x_3 = b$ is the image $\vec{x}_0 + \langle\langle \vec{v}_1, \vec{v}_2 \rangle\rangle$, called an *affine plane*, of $\langle\langle \vec{v}_1, \vec{v}_2 \rangle\rangle$ under a translation $\vec{x}_1 \rightarrow \vec{x}_0 + \vec{x}$ where \vec{x}_0 is a point lying on the plane. Since any point in space can be chosen as the origin or as the zero vector, unnecessary distinction between vector and affine spaces, except possibly for pedagogic reasons, should be emphasized or exaggerated. This is the main reason why I put the affine and linear structures of \mathbb{R}^1 , \mathbb{R}^2 , and \mathbb{R}^3 together as Part I which contains Chaps. 1, 2, and 3.

When the concepts of length and angle come into our mind, we use inner product $\langle \cdot, \cdot \rangle$ to connect both. Then the plane $a_1x_1 + a_2x_2 + a_3x_3 = b$ can be characterized as $\langle \vec{x} - \vec{x}_0, \vec{a} \rangle = 0$ where $\vec{a} = (a_1, a_2, a_3)$ is the normal vector to the plane and $\vec{x} - \vec{x}_0$ is a *vector* lying on the plane which is determined by the *points* \vec{x}_0 and \vec{x}_1 in the plane. This is Part II, the Euclidean structures of \mathbb{R}^2 and \mathbb{R}^3 , which contains Chaps. 4 and 5.

In our vivid physical world, it is difficult to realize that the parallel planes $a_1x_1 + a_2x_2 + a_3x_3 = b$ ($b \neq 0$) and $a_1x_1 + a_2x_2 + a_3x_3 = 0$ will intersect along a “line” within our sights. By central projection, it would be reasonable to imagine that they do intersect along an infinite or imaginary line l_∞ . The adjoint of l_∞ to the plane $a_1x_1 + a_2x_2 + a_3x_3 = b$ constitutes a *projective plane*. This is briefly touched in Sec. 3.8.4 of Chap. 3.

Changes of coordinates from $\vec{x} = (x_1, x_2)$ to $\vec{y} = (y_1, y_2)$ in \mathbb{R}^2 :

$$\begin{aligned} y_1 &= a_1 + a_{11}x_1 + a_{21}x_2 \\ y_2 &= a_2 + a_{12}x_1 + a_{22}x_2 \text{ or } \vec{y} = \vec{x}_0 + \vec{x} A \end{aligned}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ with } a_{11}a_{22} - a_{12}a_{21} \neq 0$$

is called an *affine transformation* and in particular, an invertible *linear transformation* if $\vec{x}_0 = (a_1, a_2) = \vec{0}$. This can be characterized as a one-to-one mapping from \mathbb{R}^2 onto \mathbb{R}^2 which preserves ratios of line segments along parallel lines (Sec. 1.3, Sec. 2.8, and Sec. 3.8). If it preserves distances between any two points, then called a *rigid* or *Euclidean motion* (Sec. 4.8 and Sec. 5.8). While $\vec{y} = \sigma(\vec{x} A)$ for any scalar $\sigma \neq 0$ maps lines onto lines on the projective plane and is called a *projective transformation* (Sec. 3.8.4). The invariants under the group (A.4) of respective transformations constitute what F. Klein called *affine*, *Euclidean* and *projective* geometries (Sec. 2.8.4, Sec. 3.8.4, Sec. 4.9 and Sec. 5.9).

As important applications of exterior products in \mathbb{R}^3 , *elliptic geometry* (Sec. 5.11) and *hyperbolic geometry* (Sec. 5.12) are introduced in the same manner as above. These two are independent of the others in the book.

Almost every text about linear algebra treats \mathbb{R}^1 trivially and obviously. Yes, really it is and hence some pieces of implicit information about \mathbb{R}^1 are usually ignored. Chapter 1 indicates that only *scalar multiplication* of a vector is just enough to describe a straight line and how the concept of *linear dependence* comes out of geometric intuition. Also, through vectorization and coordinatization of a straight line, one can realize why the abstract set \mathbb{R}^1 can be considered as standard representation of all straight lines. Changes of coordinates enable us to interpret the linear equation $y = ax + b$, $a \neq 0$, geometrically as an affine transformation preserving ratios of segment lengths. Above all, this chapter lays the foundation of inductive approach to the later chapters.

Ways of thinking and the methods adopted to realize them in Chap. 2 constitute a cornerstone for the development of the theory and a model to go after in Chap. 3 and even farther more. The fact that a point outside a given line is needed to construct a plane is algebraically equivalent to say that, in addition to scalar multiplication, the *addition of vectors* is needed in order, via concept of *linear independence* and method of *linear combination*, to go from a lower dimensional space (like straight line) to a higher one (like plane). Sec. 2.2 up to Sec. 2.4 are counterparts of Sec. 1.1 up to Sec. 1.3 and they set up the abstract set \mathbb{R}^2 as the standard two-dimensional real *vector space* and changes of coordinates in \mathbb{R}^2 . The existence of straight lines (Sec. 2.5) on \mathbb{R}^2 implicitly suggests that it is possible to discuss *vector*

and *affine* subspaces in it. Sec. 2.6 formalizes affine coordinates and introduces another useful barycentric coordinates. The important concepts of *linear (affine) transformation* and its *matrix representation* related to *bases* are main theme in Sec. 2.7 and Sec. 2.8. The geometric behaviors of *elementary matrices* considered as linear transformations are investigated in Sec. 2.7.2 along with the *factorization of a matrix* in Sec. 2.7.5 as a product of elementary ones. While, Sec. 2.7.6, Sec. 2.7.7 and Sec. 2.7.8 are concerned respectively with diagonal, Jordan and rational canonical forms of linear operators. Based on Sec. 2.7, Sec. 2.8.3 collects invariants under affine transformations and Sec. 2.8.4 introduces affine geometry in the planes. The last section Sec. 2.8.5 investigates affine invariants of quadratic curves.

Chapter 3, investigating \mathbb{R}^3 , is nothing new by nature and in content from these in Chap. 2 but is more difficult in algebraic computations and in the manipulation of geometric intuition. What should be mentioned is that, basically, only middle-school algebra is enough to handle the whole Chap. 2 but I try to transform this classical form of algebra into rudimentary ones adopted in Linear Algebra which are going to become sophisticated and formally formulated in Chap. 3.

Chapters 4 and 5 use inner product \langle, \rangle to connect concepts of length and angle. The whole theory concerned is based on the Pythagorean Theorem and *orthogonality* dominates everywhere. In addition to lines and planes, circles (Sec. 4.2), spheres (Sec. 5.2) and exterior product of vectors in \mathbb{R}^3 (Sec. 5.1) are discussed. One of the features here is that we use geometric intuition to define determinants of order 2 and 3 and to develop their algebraic operational properties (Sec. 4.3 and Sec. 5.3). An important by-product of nonnatural inner product (Sec. 4.4 and Sec. 5.4) is orthogonal matrix. Therefore, another feature is that we use geometric methods to prove *SVD for matrices* of order 2 and 3 (Sec. 4.5 and Sec. 5.5), and the *diagonalization of symmetric matrices* of order 2 and 3 (Sec. 4.7 and Sec. 5.7). Euclidean invariant and geometry are in Sec. 4.9 and Sec. 5.9. Euclidean invariants of quadratic curves and surfaces are in Sec. 4.10 and Sec. 5.10. As companions of Euclidean (also called parabolic) geometry, elliptic and hyperbolic geometries are sketched in Sec. 5.11 and Sec. 5.12, respectively.

Notations

Sections denoted by an asterisk (*) are optional and may be omitted.

A.1 means the first section in Appendix A, etc.

Sec. 1.1 means the first section in Chapter 1. So Sec. 4.3 means the third section in Chapter 4, while Sec. 5.9.1 means the first subsection of Sec. 5.9, etc.

Ex. $\langle A \rangle$ (1) of Sec. 1.1 means the first problem in Exercises $\langle A \rangle$ of Sec. 1.1, etc.

(1.1-1) means the first numbered important or remarkable facts or summarized theorem in Sec. 1.1, etc.

Fig. 3-6 means that the sixth figure in Chapter 3, etc. Fig. II-1 means the first figure in Part Two, etc. Fig. A-1 means the first figure in Appendix A; similarly for Fig. B-1, etc.

The end of a proof or an Example or a Remark is sometimes but not always marked by \square for attention.

For details, refer to *Index of Notations*.

Suggestions to the readers (how to use this book)

The materials covered in this book are rich and wide, especially in Exercises $\langle C \rangle$ and $\langle D \rangle$. It is almost impossible to cover the whole book in a single course on linear algebra when being used as a textbook for beginners.

As a textbook, the depth and wideness of materials chosen, the degree of rigorousness in proofs and how many topics of applications to be covered depend, in my opinion, mainly on the purposes designed for the course and the students' mathematical sophistication and backgrounds. Certainly, there are various such combinations and opinions. The instructors always play a central role on many occasions. The following possible choices are suggested:

- (1) For honored high school students: Chapters 1, 2, and 4 plus Exercises $\langle A \rangle$.
- (2) For freshman students: Chapters 1, 2 (up to Sec. 2.7), 3 (up to Sec. 3.7), 4 (up to Sec. 4.7 and Sec. 4.10) and/or 5 (up to Sec. 5.7 and Sec. 5.10) plus, at least, Exercises $\langle A \rangle$, in a one-academic-year three-hour-per-week course. As far as teaching order, one can adopt this original arrangement in this book, or after finishing Chapter 1, try to combine Chapters 2 and 3, 4 and 5 together according to the same titles of sections in each chapter.
- (3) For sophomore students: Just like (2) but some selected problems from Exercise $\langle B \rangle$ are contained.
- (4) For a geometric course via linear algebra: Chapters 1, 2 (Sec. 2.8), 3 (Sec. 3.8), 4 (Sec. 4.8) and 5 (Sec. 5.8~Sec. 5.12) in a one-academic-year three-hour-per-week course.