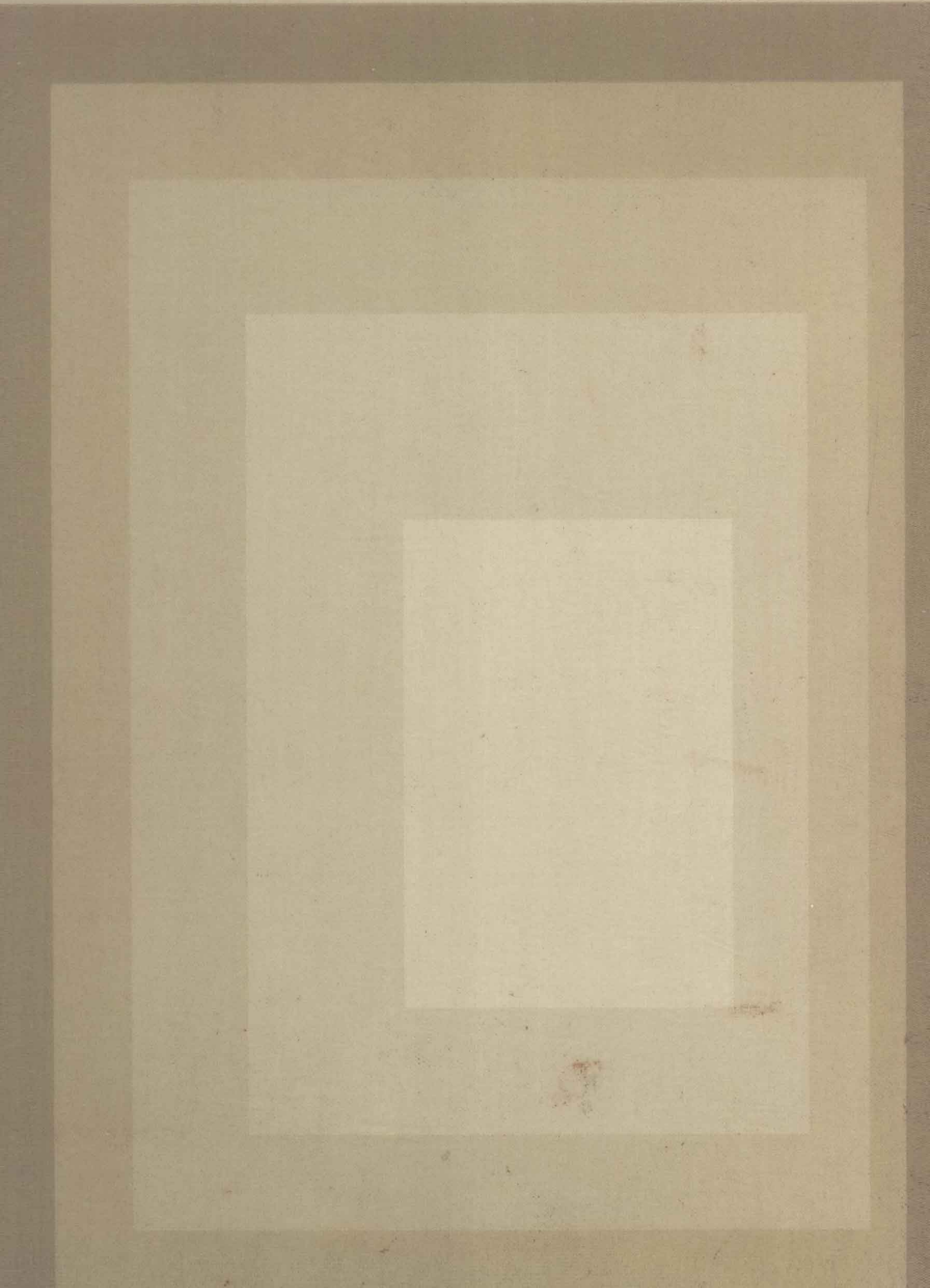
stein: **calculus** in the first three dimensions



CALCULUS in the first three dimensions

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Preface

Nowadays a beginning calculus class contains a variety of students: engineers, social and physical scientists, mathematicians, and a host of others, as well as the uncommitted. This variety prevents the teacher from treating all his students as candidates for a Ph.D. in mathematics. On the other hand, the students, whatever their eventual specialty, will need not only skill in using the calculus, but also some understanding. I believe that as long as we restrict the duration of the introductory calculus course to at most one year, we can meet the demand of the nonmathematician, and yet not delay the education of the mathematician. In a sense the challenge itself suggests a solution. After all, those who will apply the calculus should have some understanding of the definite integral, the derivative, and the relation between them, while the mathematics major should be aware of some of the applications of mathematics to other sciences.

This means that a first calculus course should have mathematical substance without encroaching on real analysis; it should have ample motivation and yet cleanly distinguish theory from application. The somewhat novel organization of this book is a response to this challenge. In Part I (Chapters 1 to 9) the student concentrates on three basic ideas: the definite integral, the derivative, and the fundamental theorem of calculus. (The definite integral is placed in Chapter 1 in order to have many pages between it and the antiderivative, with which it is too easily confused, and to alert students who may already have had a smattering of calculus.) After Chapter 6 the student may be directed along any one of several routes, perhaps into Part II or Part III. In Part II he meets such topics as the maximum and minimum of a function, Taylor's series, partial derivatives, differentiation of vectors, and Green's theorem in the plane. Part III, containing no new mathematical development, applies the techniques developed earlier (mainly those of Part I) to significant problems in the natural, social, and physical sciences.

The introductions to many concepts, such as the definite integral, the derivative, and the limit of a sequence, begin with numerical examples and exercises (whose answers are usually rounded off to three decimal places). This is done not only to make the abstract concrete, but also to compensate for a lack of down-to-earth mathematical experience in high school. In particular, both the definite integral and the derivative are preceded by four of their applications.

Such analytic geometry as is needed is developed in the text (slope of a line is defined when we examine the tangent to a curve, and the equation of a plane is obtained as an application of dot products). For convenience, an appendix on the rudiments of analytic geometry is included. Two of the other appendixes are devoted to the real numbers and functions.

Although the ε,δ terminology is introduced, and formal definitions of limits and the definite integral are presented in Chapter 3, the proofs of the basic properties of limits are left to an appendix. I do not think that there is time in a first-year calculus to develop skill with ε,δ. Rather than try to rush a heterogeneous group of students through this form of "rigor," I have chosen to include many counterexamples, and to devote more attention to the fundamental theorem of calculus.

The logarithm is considered as the inverse of a given exponential function. This approach makes more sense to most students than does the integral approach. However, the integral approach is included later (optionally) as an illustration of the fundamental theorem of calculus.

The exercises in each section are broken into two groups by the symbol ☐ ☐ ☐ The second group in each section explores fine points, extends the material, or presents more difficulties than the first group, which, it should be emphasized, contains more than enough routine problems to give the students an opportunity to develop both skill and understanding. Answers accompany many exercises in the first group; the Teacher's Manual contains, in addition to answers to exercises in the first group, solutions to exercises in the second group.

It seems to me that a student of the calculus should become familiar with a handbook of mathematical tables, which includes such useful items as the decimal expansion of $1/n$ and a review of trigonometry. Hence the text contains only the briefest tables—enough to point out the behavior of the principal functions met in the calculus.

The Teacher's Manual discusses the use of the text in greater detail, as well as its relation to CUPM recommendations.

The criticisms of two mathematicians, Raymond A. Barnett of Oakland City College and George N. Raney of the University of Connecticut, led to most of the differences between the first and final drafts of this book. Edwin H. Spanier of the University of California at Berkeley and William Simons of Oregon State University influenced the exposition in several chapters. My colleagues in the mathematics department and several other departments on my campus advised me on many problems that arose during the writing.

Furthermore I wish to thank the *American Mathematical Monthly*, published by the Mathematical Association of America, for permission to quote from its pages. The many reviews of calculus texts and pedagogical articles published there in the past forty years—surprisingly consistent in their disappointments and suggestions—helped shape this book.

These acknowledgements would be incomplete without my expression of appreciation to my wife and children, who sustained my morale during the many months of writing and rewriting.

Sherman K. Stein

To my parents

Harry Stein and Fannie Kopald Stein

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PART I

The core of the calculus

If an object moves at a constant speed, then we can find the distance it travels by simply multiplying its speed by the duration of its journey. But if the speed varies from instant to instant, how can we then compute the length of its journey? Four questions of this type lead us in Chapter 1 to the definite integral, *one of the two basic concepts of the calculus.*

The question can be turned around: If we happen to know how far an object, moving with a varying speed, travels during any interval of time (for instance, a rock drops $16t^2$ feet in the first t seconds of a free fall), then how can we find its speed at any instant? Such questions, posed in Chapter 2, introduce the derivative, *the other basic concept of the calculus.*

In Chapters 3 through 6 we develop tools for answering both questions. Chapters 7 through 9 offer an opportunity to develop skill in applying these tools in a variety of situations.