Concepts and Applications of Continuum Mechanics

Edited by **Derek Pearce**

NY RESEARCH PRESS
New York

Published by NY Research Press,
23 West, 55th Street, Suite 816,
New York, NY 10019, USA
www.nyresearchpress.com

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International Standard Book Number: 978-1-63238-091-3 (Hardback)

Printed in China.

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Preface

Covering every aspect of Continuum Mechanics, this book brilliantly elucidates its concepts and applications. Continuum mechanics is the base of Applied Mechanics. There are a number of books on Continuum Mechanics emphasizing on the macro-scale mechanical conduct of materials. Unlike traditional Continuum Mechanics books, this book provides synopsis on the developments in some specific areas of Continuum Mechanics. This book focuses primarily on the applications aspects. Energy materials and systems i.e. fuel cell materials and electrodes, substance deportation and mechanical response/deformation of plates, pipelines etc. have been covered under the applications described in this book. Researchers from different fields will benefit from reading the mechanics approach to solve engineering problems.

After months of intensive research and writing, this book is the end result of all who devoted their time and efforts in the initiation and progress of this book. It will surely be a source of reference in enhancing the required knowledge of the new developments in the area. During the course of developing this book, certain measures such as accuracy, authenticity and research focused analytical studies were given preference in order to produce a comprehensive book in the area of study.

This book would not have been possible without the efforts of the authors and the publisher. I extend my sincere thanks to them. Secondly, I express my gratitude to my family and well-wishers. And most importantly, I thank my students for constantly expressing their willingness and curiosity in enhancing their knowledge in the field, which encourages me to take up further research projects for the advancement of the area.

Editor

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Spencer Operator and Applications: From Continuum Mechanics to Mathematical Physics

J.F. Pommaret
CERMICS, Ecole Nationale des Ponts et Chaussées, France

1. Introduction

Let us revisit briefly the foundation of n-dimensional elasticity theory as it can be found today in any textbook, restricting our study to $n = 2$ for simplicity. If $x = (x^1, x^2)$ is a point in the plane and $\xi = (\xi^1(x), \xi^2(x))$ is the displacement vector, lowering the indices by means of the Euclidean metric, we may introduce the "small" deformation tensor $\epsilon = (\epsilon_{ij} = \epsilon_{ji} = (1/2)(\partial_i\xi_j + \partial_j\xi_i))$ with $n(n+1)/2 = 3$ (independent) *components* $(\epsilon_{11}, \epsilon_{12} = \epsilon_{21}, \epsilon_{22})$. If we study a part of a deformed body, for example a thin elastic plane sheet, by means of a variational principle, we may introduce the local density of free energy $\varphi(\epsilon) = \varphi(\epsilon_{ij}|i \leq j) = \varphi(\epsilon_{11}, \epsilon_{12}, \epsilon_{22})$ and vary the total free energy $F = \int \varphi(\epsilon)dx$ with $dx = dx^1 \wedge dx^2$ by introducing $\sigma^{ij} = \partial\varphi/\partial\epsilon_{ij}$ for $i \leq j$ in order to obtain $\delta F = \int(\sigma^{11}\delta\epsilon_{11} + \sigma^{12}\delta\epsilon_{12} + \sigma^{22}\delta\epsilon_{22})dx$. Accordingly, the "decision" to define the stress tensor σ by a symmetric matrix with $\sigma^{12} = \sigma^{21}$ is purely artificial within such a variational principle. Indeed, the usual Cauchy device (1828) assumes that each element of a boundary surface is acted on by a surface density of force $\vec{\sigma}$ with a linear dependence $\vec{\sigma} = (\sigma^{ir}(x)n_r)$ on the outward normal unit vector $\vec{n} = (n_r)$ and does not make any assumption on the stress tensor. It is only by an equilibrium of forces and couples, namely the well known *phenomenological static torsor equilibrium*, that one can "prove" the symmetry of σ. However, even if we assume this symmetry, *we now need the different summation* $\sigma^{ij}\delta\epsilon_{ij} = \sigma^{11}\delta\epsilon_{11} + 2\sigma^{12}\delta\epsilon_{12} + \sigma^{22}\delta\epsilon_{22} = \sigma^{ir}\partial_r\delta\xi_i$. An integration by parts and a change of sign produce the volume integral $\int(\partial_r\sigma^{ir})\delta\xi_i dx$ leading to the stress equations $\partial_r\sigma^{ir} = 0$. *The classical approach to elasticity theory, based on invariant theory with respect to the group of rigid motions, cannot therefore describe equilibrium of torsors by means of a variational principle where the proper torsor concept is totally lacking.*

There is another equivalent procedure dealing with a *variational calculus with constraint*. Indeed, as we shall see in Section 7, the deformation tensor is not any symmetric tensor as it must satisfy $n^2(n^2-1)/12$ compatibility conditions (CC), that is only $\partial_{22}\epsilon_{11} + \partial_{11}\epsilon_{22} - 2\partial_{12}\epsilon_{12} = 0$ when $n = 2$. In this case, introducing the *Lagrange multiplier* $-\phi$ for convenience, *we have to vary* $\int(\varphi(\epsilon) - \phi(\partial_{22}\epsilon_{11} + \partial_{11}\epsilon_{22} - 2\partial_{12}\epsilon_{12}))dx$ *for an arbitrary* ϵ. A double integration by parts now provides the parametrization $\sigma^{11} = \partial_{22}\phi, \sigma^{12} = \sigma^{21} = -\partial_{12}\phi, \sigma^{22} = \partial_{11}\phi$ of the stress equations by means of the Airy function ϕ and the *formal adjoint* of the CC, *on the condition to observe that we have in fact* $2\sigma^{12} = -2\partial_{12}\phi$ as another way to understand the deep meaning of the factor "2" in the summation. In arbitrary dimension, it just remains to notice that the above compatibility conditions are nothing else but the linearized Riemann tensor in Riemannan geometry, a crucial mathematical tool in the theory of general relativity.

It follows that the only possibility to revisit the foundations of engineering and mathematical physics is to use new mathematical methods, namely the theory of systems of partial differential equations and Lie pseudogroups developped by D.C. Spencer and coworkers during the period 1960-1975. In particular, Spencer invented the first order operator now wearing his name in order to bring in a canonical way the formal study of systems of ordinary differential (OD) or partial differential (PD) equations to that of equivalent first order systems. However, despite its importance, the *Spencer operator* is rarely used in mathematics today and, up to our knowledge, has never been used in engineering or mathematical physics. The main reason for such a situation is that the existing papers, largely based on hand-written lecture notes given by Spencer to his students (the author was among them in 1969) are quite technical and the problem also lies in the only "accessible" book "Lie equations" he published in 1972 with A. Kumpera. Indeed, the reader can easily check by himself that *the core of this book has nothing to do with its introduction* recalling known differential geometric concepts on which most of physics is based today.

The first and technical purpose of this chapter, an extended version of a lecture at the second workshop on Differential Equations by Algebraic Methods (DEAM2, february 9-11, 2011, Linz, Austria), is to recall briefly its definition, both in the framework of systems of linear ordinary or partial differential equations and in the framework of differential modules. The local theory of Lie pseudogroups and the corresponding non-linear framework are also presented for the first time in a rather elementary manner though it is a difficult task.

The second and central purpose is to prove that the use of the Spencer operator constitutes the *common secret* of the three following famous books published about at the same time in the beginning of the last century, though they do not seem to have anything in common at first sight as they are successively dealing with the foundations of elasticity theory, commutative algebra, electromagnetism (EM) and general relativity (GR):

[C] E. and F. COSSERAT: "Théorie des Corps Déformables", Hermann, Paris, 1909.
[M] F.S. MACAULAY: "The Algebraic Theory of Modular Systems", Cambridge, 1916.
[W] H. WEYL: "Space, Time, Matter", Springer, Berlin, 1918 (1922, 1958; Dover, 1952).

Meanwhile we shall point out the striking importance of the second book for studying *identifiability* in control theory. We shall also obtain from the previous results the group theoretical unification of finite elements in engineering sciences (elasticity, heat, electromagnetism), solving the *torsor problem* and recovering in a purely mathematical way known *field-matter coupling phenomena* (piezzoelectricity, photoelasticity, streaming birefringence, viscosity, ...).

As a byproduct and though disturbing it may be, the third and perhaps essential purpose is to prove that *these unavoidable new differential and homological methods contradict the existing mathematical foundations of both engineering (continuum mechanics, electromagnetism) and mathematical (gauge theory, general relativity) physics.*

Many explicit examples will illustate this chapter which is deliberately written in a rather self-contained way to be accessible to a large audience, which does not mean that it is elementary in view of the number of new concepts that must be patched together. However, the reader must never forget that *each formula* appearing in this new general framework has been used explicitly or implicitly in [C], [M] and [W] for a mechanical, mathematical or physical purpose.

2. From Lie groups to Lie pseudogroups

Evariste Galois (1811-1832) introduced the word "*group*" for the first time in 1830. Then the group concept slowly passed from algebra (groups of permutations) to geometry (groups of transformations). It is only in 1880 that Sophus Lie (1842-1899) studied the groups of transformations depending on a finite number of parameters and now called *Lie groups of transformations*. Let X be a manifold with local coordinates $x = (x^1, ..., x^n)$ and G be a *Lie group*, that is another manifold with local coordinates $a = (a^1, ..., a^p)$ called *parameters* with a *composition* $G \times G \to G : (a,b) \to ab$, an *inverse* $G \to G : a \to a^{-1}$ and an *identity* $e \in G$ satisfying:

$$(ab)c = a(bc) = abc, \qquad aa^{-1} = a^{-1}a = e, \qquad ae = ea = a, \qquad \forall a,b,c \in G$$

Definition 2.1. *G is said to act on X if there is a map $X \times G \to X : (x,a) \to y = ax = f(x,a)$ such that $(ab)x = a(bx) = abx, \forall a,b \in G, \forall x \in X$ and, for simplifying the notations, we shall use global notations even if only local actions are existing. The set $G_x = \{a \in G \mid ax = x\}$ is called the isotropy subgroup of G at $x \in X$. The action is said to be effective if $ax = x, \forall x \in X \Rightarrow a = e$. A subset $S \subset X$ is said to be invariant under the action of G if $aS \subset S, \forall a \in G$ and the orbit of $x \in X$ is the invariant subset $Gx = \{ax \mid a \in G\} \subset X$. If G acts on two manifolds X and Y, a map $f : X \to Y$ is said to be equivariant if $f(ax) = af(x), \forall x \in X, \forall a \in G$.*

For reasons that will become clear later on, it is often convenient to introduce the *graph* $X \times G \to X \times X : (x,a) \to (x, y = ax)$ of the action. In the product $X \times X$, the first factor is called the *source* while the second factor is called the *target*.

Definition 2.2. *The action is said to be free if the graph is injective and transitive if the graph is surjective. The action is said to be simply transitive if the graph is an isomorphism and X is said to be a principal homogeneous space (PHS) for G.*

In order to fix the notations, we quote without any proof the "*Three Fundamental Theorems of Lie*" that will be of constant use in the sequel ([26]):

First fundamental theorem: The orbits $x = f(x_0, a)$ satisfy the system of PD equations $\partial x^i / \partial a^\sigma = \theta^i_\rho(x)\omega^\rho_\sigma(a)$ with $det(\omega) \neq 0$. The vector fields $\theta_\rho = \theta^i_\rho(x)\partial_i$ are called *infinitesimal generators* of the action and are linearly independent over the constants when the action is effective.

If X is a manifold, we denote as usual by $T = T(X)$ the *tangent bundle* of X, by $T^* = T^*(X)$ the *cotangent bundle*, by $\wedge^r T^*$ the *bundle of r-forms* and by $S_q T^*$ the *bundle of q-symmetric tensors*. More generally, let $\mathcal{E}$ be a *fibered manifold*, that is a manifold with local coordinates (x^i, y^k) for $i = 1, ..., n$ and $k = 1, ..., m$ simply denoted by (x, y), *projection* $\pi : \mathcal{E} \to X : (x,y) \to (x)$ and changes of local coordinates $\bar{x} = \varphi(x), \bar{y} = \psi(x,y)$. If $\mathcal{E}$ and $\mathcal{F}$ are two fibered manifolds over X with respective local coordinates (x,y) and (x,z), we denote by $\mathcal{E} \times_X \mathcal{F}$ the *fibered product* of $\mathcal{E}$ and $\mathcal{F}$ over X as the new fibered manifold over X with local coordinates (x,y,z). We denote by $f : X \to \mathcal{E} : (x) \to (x, y = f(x))$ a global *section* of $\mathcal{E}$, that is a map such that $\pi \circ f = id_X$ but local sections over an open set $U \subset X$ may also be considered when needed. Under a change of coordinates, a section transforms like $\bar{f}(\varphi(x)) = \psi(x, f(x))$ and the derivatives transform like:

$$\frac{\partial \bar{f}^l}{\partial \bar{x}^r}(\varphi(x))\partial_i \varphi^r(x) = \frac{\partial \psi^l}{\partial x^i}(x, f(x)) + \frac{\partial \psi^l}{\partial y^k}(x, f(x))\partial_i f^k(x)$$

We may introduce new coordinates (x^i, y^k, y^k_i) transforming like:

$$\bar{y}^l_r \partial_i \varphi^r(x) = \frac{\partial \psi^l}{\partial x^i}(x,y) + \frac{\partial \psi^l}{\partial y^k}(x,y) y^k_i$$

We shall denote by $J_q(\mathcal{E})$ the *q-jet bundle* of $\mathcal{E}$ with local coordinates $(x^i, y^k, y^k_i, y^k_{ij}, ...) = (x, y_q)$ called *jet coordinates* and sections $f_q : (x) \rightarrow (x, f^k(x), f^k_i(x), f^k_{ij}(x), ...) = (x, f_q(x))$ transforming like the sections $j_q(f) : (x) \rightarrow (x, f^k(x), \partial_i f^k(x), \partial_{ij} f^k(x), ...) = (x, j_q(f)(x))$ where both f_q and $j_q(f)$ are over the section f of $\mathcal{E}$. Of course $J_q(\mathcal{E})$ is a fibered manifold over X with projection π_q while $J_{q+r}(\mathcal{E})$ is a fibered manifold over $J_q(\mathcal{E})$ with projection $\pi^{q+r}_q, \forall r \geq 0$.

Definition 2.3. *A system of order q on $\mathcal{E}$ is a fibered submanifold $\mathcal{R}_q \subset J_q(\mathcal{E})$ and a solution of $\mathcal{R}_q$ is a section f of $\mathcal{E}$ such that $j_q(f)$ is a section of $\mathcal{R}_q$.*

Definition 2.4. *When the changes of coordinates have the linear form $\bar{x} = \varphi(x), \bar{y} = A(x)y$, we say that $\mathcal{E}$ is a vector bundle over X and denote for simplicity a vector bundle and its set of sections by the same capital letter E. When the changes of coordinates have the form $\bar{x} = \varphi(x), \bar{y} = A(x)y + B(x)$ we say that $\mathcal{E}$ is an affine bundle over X and we define the associated vector bundle E over X by the local coordinates (x, v) changing like $\bar{x} = \varphi(x), \bar{v} = A(x)v$.*

Definition 2.5. *If the tangent bundle $T(\mathcal{E})$ has local coordinates (x, y, u, v) changing like $\bar{u}^j = \partial_i \varphi^j(x) u^i, \bar{v}^l = \frac{\partial \psi^l}{\partial x^i}(x,y) u^i + \frac{\partial \psi^l}{\partial y^k}(x,y) v^k$, we may introduce the vertical bundle $V(\mathcal{E}) \subset T(\mathcal{E})$ as a vector bundle over $\mathcal{E}$ with local coordinates (x, y, v) obtained by setting $u = 0$ and changes $\bar{v}^l = \frac{\partial \psi^l}{\partial y^k}(x,y) v^k$. Of course, when $\mathcal{E}$ is an affine bundle with associated vector bundle E over X, we have $V(\mathcal{E}) = \mathcal{E} \times_X E$.*

For a later use, if $\mathcal{E}$ is a fibered manifold over X and f is a section of $\mathcal{E}$, we denote by $f^{-1}(V(\mathcal{E}))$ the *reciprocal image* of $V(\mathcal{E})$ by f as the vector bundle over X obtained when replacing (x, y, v) by $(x, f(x), v)$ in each chart. It is important to notice in variational calculus that a *variation* δf of f is such that $\delta f(x)$, as a vertical vector field not necessary "small", is a section of this vector bundle and that $(f, \delta f)$ is nothing else than a section of $V(\mathcal{E})$ over X.

We now recall a few basic geometric concepts that will be constantly used. First of all, if $\xi, \eta \in T$, we define their *bracket* $[\xi, \eta] \in T$ by the local formula $([\xi, \eta])^i(x) = \xi^r(x)\partial_r \eta^i(x) - \eta^s(x)\partial_s \xi^i(x)$ leading to the *Jacobi identity* $[\xi, [\eta, \zeta]] + [\eta, [\zeta, \xi]] + [\zeta, [\xi, \eta]] = 0, \forall \xi, \eta, \zeta \in T$ allowing to define a *Lie algebra* and to the useful formula $[T(f)(\xi), T(f)(\eta)] = T(f)([\xi, \eta])$ where $T(f) : T(X) \rightarrow T(Y)$ is the tangent mapping of a map $f : X \rightarrow Y$.

Second fundamental theorem: If $\theta_1, ..., \theta_p$ are the infinitesimal generators of the effective action of a lie group G on X, then $[\theta_\rho, \theta_\sigma] = c^\tau_{\rho\sigma}\theta_\tau$ where the $c^\tau_{\rho\sigma}$ are the *structure constants* of a Lie algebra of vector fields which can be identified with $\mathcal{G} = T_e(G)$.

When $I = \{i_1 < ... < i_r\}$ is a multi-index, we may set $dx^I = dx^{i_1} \wedge ... \wedge dx^{i_r}$ for describing $\wedge^r T^*$ and introduce the *exterior derivative* $d : \wedge^r T^* \rightarrow \wedge^{r+1} T^* : \omega = \omega_I dx^I \rightarrow d\omega = \partial_i \omega_I dx^i \wedge dx^I$ with $d^2 = d \circ d \equiv 0$ in the *Poincaré sequence*:

$$\wedge^0 T^* \xrightarrow{d} \wedge^1 T^* \xrightarrow{d} \wedge^2 T^* \xrightarrow{d} ... \xrightarrow{d} \wedge^n T^* \longrightarrow 0$$

The *Lie derivative* of an r-form with respect to a vector field $\xi \in T$ is the linear first order operator $\mathcal{L}(\xi)$ linearly depending on $j_1(\xi)$ and uniquely defined by the following three properties:

1. $\mathcal{L}(\xi)f = \xi.f = \xi^i \partial_i f, \forall f \in \wedge^0 T^* = C^\infty(X)$.
2. $\mathcal{L}(\xi)d = d\mathcal{L}(\xi)$.
3. $\mathcal{L}(\xi)(\alpha \wedge \beta) = (\mathcal{L}(\xi)\alpha) \wedge \beta + \alpha \wedge (\mathcal{L}(\xi)\beta), \forall \alpha, \beta \in \wedge T^*$.

It can be proved that $\mathcal{L}(\xi) = i(\xi)d + di(\xi)$ where $i(\xi)$ is the *interior multiplication* $(i(\xi)\omega)_{i_1 \dots i_r} = \xi^i \omega_{ii_1 \dots i_r}$ and that $[\mathcal{L}(\xi), \mathcal{L}(\eta)] = \mathcal{L}(\xi) \circ \mathcal{L}(\eta) - \mathcal{L}(\eta) \circ \mathcal{L}(\xi) = \mathcal{L}([\xi, \eta]), \forall \xi, \eta \in T$.

Using crossed-derivatives in the PD equations of the First Fundamental Theorem and introducing the family of 1-forms $\omega^\tau = \omega^\tau_\sigma(a) da^\sigma$ both with the matrix $\alpha = \omega^{-1}$ of right invariant vector fields, we obtain the *compatibility conditions* (CC) expressed by the following corollary where one must care about the sign used:

Corollary 2.1. *One has the Maurer-Cartan (MC) equations* $d\omega^\tau + c^\tau_{\rho\sigma} \omega^\rho \wedge \omega^\sigma = 0$ *or the equivalent relations* $[\alpha_\rho, \alpha_\sigma] = c^\tau_{\rho\sigma} \alpha_\tau$.

Applying d to the MC equations and substituting, we obtain the *integrability conditions* (IC):

Third fundamental theorem For any Lie algebra $\mathcal{G}$ defined by structure constants satisfying :

$$c^\tau_{\rho\sigma} + c^\tau_{\sigma\rho} = 0, \qquad c^\lambda_{\mu\rho} c^\mu_{\sigma\tau} + c^\lambda_{\mu\sigma} c^\mu_{\tau\rho} + c^\lambda_{\mu\tau} c^\mu_{\rho\sigma} = 0$$

one can construct an analytic group G such that $\mathcal{G} = T_e(G)$.

Example 2.1. *Considering the affine group of transformations of the real line* $y = a^1 x + a^2$, *we obtain* $\theta_1 = x\partial_x, \theta_2 = \partial_x \Rightarrow [\theta_1, \theta_2] = -\theta_2$ *and thus* $\omega^1 = (1/a^1) da^1, \omega^2 = da^2 - (a^2/a^1) da^1 \Rightarrow d\omega^1 = 0, d\omega^2 - \omega^1 \wedge \omega^2 = 0 \Leftrightarrow [\alpha_1, \alpha_2] = -\alpha_2$ *with* $\alpha_1 = a^1 \partial_1 + a^2 \partial_2, \alpha_2 = \partial_2$.

Only ten years later Lie discovered that the Lie groups of transformations are only particular examples of a wider class of groups of transformations along the following definition where $aut(X)$ denotes the group of all local diffeomorphisms of X:

Definition 2.6. *A Lie pseudogroup of transformations* $\Gamma \subset aut(X)$ *is a group of transformations solutions of a system of OD or PD equations such that, if* $y = f(x)$ *and* $z = g(y)$ *are two solutions, called finite transformations, that can be composed, then* $z = g \circ f(x) = h(x)$ *and* $x = f^{-1}(y) = g(y)$ *are also solutions while* $y = x$ *is a solution.*

The underlying system may be nonlinear and of high order as we shall see later on. We shall speak of an *algebraic pseudogroup* when the system is defined by *differential polynomials* that is polynomials in the derivatives. In the case of Lie groups of transformations the system is obtained by differentiating the action law $y = f(x, a)$ with respect to x as many times as necessary in order to eliminate the parameters. Looking for transformations "close" to the identity, that is setting $y = x + t\xi(x) + \dots$ when $t \ll 1$ is a small constant parameter and passing to the limit $t \to 0$, we may linearize the above nonlinear *system of finite Lie equations* in order to obtain a linear *system of infinitesimal Lie equations* of the same order for vector fields. Such a system has the property that, if ξ, η are two solutions, then $[\xi, \eta]$ is also a solution. Accordingly, the set $\Theta \subset T$ of solutions of this new system satifies $[\Theta, \Theta] \subset \Theta$ and can therefore be considered as the Lie algebra of Γ.

Though the collected works of Lie have been published by his student F. Engel at the end of the 19^{th} century, these ideas did not attract a large audience because the fashion in Europe was analysis. Accordingly, at the beginning of the 20^{th} century and for more than fifty years, only two frenchmen tried to go further in the direction pioneered by Lie, namely Elie Cartan (1869-1951) who is quite famous today and Ernest Vessiot (1865-1952) who is almost ignored today, each one deliberately ignoring the other during his life for a precise reason that we now explain with details as it will surprisingly constitute the heart of this chapter. (The author is indebted to Prof. Maurice Janet (1888-1983), who was a personal friend of Vessiot, for the many documents he gave him, partly published in [25]). Roughly, the idea of many people at that time was to extend the work of Galois along the following scheme of increasing difficulty:

1) *Galois theory* ([34]): Algebraic equations and permutation groups.
2) *Picard-Vessiot theory* ([17]): OD equations and Lie groups.
3) *Differential Galois theory* ([24],[37]): PD equations and Lie pseudogroups.

In 1898 Jules Drach (1871-1941) got and published a thesis ([9]) with a jury made by Gaston Darboux, Emile Picard and Henri Poincaré, the best leading mathematicians of that time. However, despite the fact that it contains ideas quite in advance on his time such as the concept of a "differential field" only introduced by J.-F. Ritt in 1930 ([31]), the jury did not notice that the main central result was wrong: Cartan found the counterexamples, Vessiot recognized the mistake, Paul Painlevé told it to Picard who agreed but Drach never wanted to acknowledge this fact and was supported by the influent Emile Borel. As a byproduct, everybody flew out of this "affair", never touching again the Galois theory. After publishing a prize-winning paper in 1904 where he discovered for the first time that the differential Galois theory must be a theory of (irreducible) PHS for algebraic pseudogroups, Vessiot remained alone, trying during thirty years to correct the mistake of Drach that we finally corrected in 1983 ([24]).

3. Cartan versus Vessiot : The structure equations

We study first the work of Cartan which can be divided into two parts. The first part, for which he invented exterior calculus, may be considered as a tentative to extend the MC equations from Lie groups to Lie pseudogroups. The idea for that is to consider the system of order q and its *prolongations* obtained by differentiating the equations as a way to know certain derivatives called *principal* from all the other arbitrary ones called *parametric* in the sense of Janet ([13]). Replacing the derivatives by jet coordinates, we may try to copy the procedure leading to the MC equations by using a kind of "composition" and "inverse" on the jet coordinates. For example, coming back to the last definition, we get successively:

$$\frac{\partial h}{\partial x} = \frac{\partial g}{\partial y}\frac{\partial f}{\partial x}, \qquad \frac{\partial^2 h}{\partial x^2} = \frac{\partial^2 g}{\partial y^2}\frac{\partial f}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial^2 f}{\partial x^2}, \ldots$$

Now if $g = f^{-1}$ then $g \circ f = id$ and thus $\frac{\partial g}{\partial y}\frac{\partial f}{\partial x} = 1, \ldots$ while the new identity $id_q = j_q(id)$ is made by the successive derivatives of $y = x$, namely $(1, 0, 0, \ldots)$. These *awfully complicated computations* bring a lot of structure constants and have been so much superseded by the work of Donald C. Spencer (1912-2001) ([11],[12],[18],[33]) that, in our opinion based on thirty years of explicit computations, this tentative has only been used for classification problems and is not useful for applications compared to the results of the next sections. In a single concluding sentence, *Cartan has not been able to "go down" to the base manifold X while Spencer did succeed fifty years later.*

We shall now describe the second part with more details as it has been (and still is !) the crucial tool used in both engineering (analytical and continuum mechanics) and mathematical (gauge theory and general relativity) physics in an absolutely contradictory manner. We shall try to use the least amount of mathematics in order to prepare the reader for the results presented in the next sections. For this let us start with an elementary experiment that will link at once continuum mechanics and gauge theory in an unusual way. Let us put a thin elastic rectilinear rubber band along the x axis on a flat table and translate it along itself. The band will remain identical as no deformation can be produced by this constant translation. However, if we move each point continuously along the same direction but in a point depending way, for example fixing one end and pulling on the other end, there will be of course a deformation of the elastic band according to the Hooke law. It remains to notice that a constant translation can be written in the form $y = x + a^2$ as in Example 2.1 while a point depending translation can be written in the form $y = x + a^2(x)$ which is written in any textbook of continuum mechanics in the form $y = x + \xi(x)$ by introducing the *displacement vector* ξ. However nobody could even imagine to make a^1 also point depending and to consider $y = a^1(x)x + a^2(x)$ as we shall do in Example 7.2.We also notice that the movement of a rigid body in space may be written in the form $y = a(t)x + b(t)$ where now $a(t)$ is a time depending orthogonal matrix and $b(t)$ is a time depending vector. What makes all the difference between the two examples is that the group is *acting* on x in the first but *not acting* on t in the second. Finally, a point depending rotation or dilatation is not accessible to intuition and the general theory must be done in the following manner.

If X is a manifold and G is a lie group *not acting necessarily* on X, let us consider maps $a : X \to G : (x) \to (a(x))$ or equivalently sections of the trivial (principal) bundle $X \times G$ over X. If $x + dx$ is a point of X close to x, then $T(a)$ will provide a point $a + da = a + \frac{\partial a}{\partial x}dx$ close to a on G. We may bring a back to e on G by acting on a with a^{-1}, *either on the left or on the right*, getting therefore a 1-form $a^{-1}da = A$ or $daa^{-1} = B$. As $aa^{-1} = e$ we also get $daa^{-1} = -ada^{-1} = -b^{-1}db$ if we set $b = a^{-1}$ as a way to link A with B. When there is an action $y = ax$, we have $x = a^{-1}y = by$ and thus $dy = dax = daa^{-1}y$, a result leading through the First Fundamental Theorem of Lie to the equivalent formulas:

$$a^{-1}da = A = (A_i^\tau(x)dx^i = -\omega_\sigma^\tau(b(x))\partial_i b^\sigma(x)dx^i)$$

$$daa^{-1} = B = (B_i^\tau(x)dx^i = \omega_\sigma^\tau(a(x))\partial_i a^\sigma(x)dx^i)$$

Introducing the induced bracket $[A, A](\xi, \eta) = [A(\xi), A(\eta)] \in \mathcal{G}, \forall \xi, \eta \in T$, we may define the 2-form $dA - [A, A] = F \in \wedge^2 T^* \otimes \mathcal{G}$ by the local formula (care to the sign):

$$\partial_i A_j^\tau(x) - \partial_j A_i^\tau(x) - c_{\rho\sigma}^\tau A_i^\rho(x) A_j^\sigma(x) = F_{ij}^\tau(x)$$

and obtain from the second fundamental theorem:

Theorem 3.1. *There is a nonlinear gauge sequence:*

$$\begin{array}{ccccc} X \times G & \longrightarrow & T^* \otimes \mathcal{G} & \stackrel{MC}{\longrightarrow} & \wedge^2 T^* \otimes \mathcal{G} \\ a & \longrightarrow & a^{-1}da = A & \longrightarrow & dA - [A, A] = F \end{array}$$

Choosing a "close" to e, that is $a(x) = e + t\lambda(x) + ...$ and linearizing as usual, we obtain the linear operator $d : \wedge^0 T^* \otimes \mathcal{G} \to \wedge^1 T^* \otimes \mathcal{G} : (\lambda^\tau(x)) \to (\partial_i \lambda^\tau(x))$ leading to:

Corollary 3.1. *There is a linear gauge sequence:*

$$\wedge^0 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^1 T^* \otimes \mathcal{G} \xrightarrow{d} \wedge^2 T^* \otimes \mathcal{G} \xrightarrow{d} ... \xrightarrow{d} \wedge^n T^* \otimes \mathcal{G} \longrightarrow 0$$

which is the tensor product by $\mathcal{G}$ of the Poincaré sequence:

Remark 3.1. *When the physicists C.N. Yang and R.L. Mills created (non-abelian) gauge theory in 1954 ([38],[39]), their work was based on these results which were the only ones known at that time, the best mathematical reference being the well known book by S. Kobayashi and K. Nomizu ([15]). It follows that the only possibility to describe elecromagnetism (EM) within this framework was to call A the Yang-Mills potential and F the Yang-Mills field (a reason for choosing such notations) on the condition to have $dim(G) = 1$ in the abelian situation $c = 0$ and to use a Lagrangian depending on $F = dA - [A, A]$ in the general case. Accordingly the idea was to select the unitary group $U(1)$, namely the unit circle in the complex plane with Lie algebra the tangent line to this circle at the unity $(1, 0)$. It is however important to notice that the resulting Maxwell equations $dF = 0$ have no equivalent in the non-abelian case $c \neq 0$.*

Just before Albert Einstein visited Paris in 1922, Cartan published many short Notes ([5]) announcing long papers ([6]) where he selected G to be the Lie group involved in the Poincaré (conformal) group of space-time preserving (up to a function factor) the Minkowski metric $\omega = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2$ with $x^4 = ct$ where c is the speed of light. In the first case F is decomposed into two parts, the *torsion* as a 2-form with value in translations on one side and the *curvature* as a 2-form with value in rotations on the other side. This result was looking coherent *at first sight* with the Hilbert variational scheme of general relativity (GR) introduced by Einstein in 1915 ([21],[38]) and leading to a Lagrangian depending on $F = dA - [A, A]$ as in the last remark.

In the meantime, Poincaré developped an invariant variational calculus ([22]) which has been used again without any quotation, successively by G. Birkhoff and V. Arnold (compare [4], 205-216 with [2], 326, Th 2.1). A particular case is well known by any student in the analytical mechanics of rigid bodies. Indeed, using standard notations, the movement of a rigid body is described in a fixed Cartesian frame by the formula $x(t) = a(t)x_0 + b(t)$ where $a(t)$ is a 3×3 time dependent orthogonal matrix (rotation) and $b(t)$ a time depending vector (translation) as we already said. Differentiating with respect to time by using a dot, the *absolute speed* is $v = \dot{x}(t) = \dot{a}(t)x_0 + \dot{b}(t)$ and we obtain the *relative speed* $a^{-1}(t)v = a^{-1}(t)\dot{a}(t)x_0 + a^{-1}(t)\dot{b}(t)$ by projection in a frame fixed in the body. Having in mind Example 2.1, it must be noticed that the so-called *Eulerian speed* $v = v(x, t) = \dot{a}a^{-1}x + \dot{b} - \dot{a}a^{-1}b$ only depends on the 1-form $B = (\dot{a}a^{-1}, \dot{b} - \dot{a}a^{-1}b)$. The Lagrangian (kinetic energy in this case) is thus a quadratic function of the 1-form $A = (a^{-1}\dot{a}, a^{-1}\dot{b})$ where $a^{-1}\dot{a}$ is a 3×3 skew symmetric time depending matrix. Hence, "*surprisingly*", this result is not coherent at all with EM where the Lagrangian is the quadratic expression $(\epsilon/2)E^2 - (1/2\mu)B^2$ because the electric field $\vec{E}$ and the magnetic field $\vec{B}$ are combined in the EM field F as a 2-form satisfying the first set of Maxwell equations $dF = 0$. The dielectric constant ϵ and the magnetic constant μ are leading to the electric induction $\vec{D} = \epsilon\vec{E}$ and the magnetic induction $\vec{H} = (1/\mu)\vec{B}$ in the second set of Maxwell equations. In view of the existence of well known field-matter couplings such as piezoelectricity and photoelasticity that will be described later on, such a situation is contradictory as it should lead to put on equal footing 1-forms and 2-forms contrary to any unifying mathematical scheme but no other substitute could have been provided at that time.

Let us now turn to the other way proposed by Vessiot in 1903 ([36]) and 1904 ([37]). Our purpose is only to sketch the main results that we have obtained in many books ([23-26], we do not know other references) and to illustrate them by a series of specific examples, asking the reader to imagine any link with what has been said.

1. If $\mathcal{E} = X \times X$, we shall denote by $\Pi_q = \Pi_q(X,X)$ the open subfibered manifold of $J_q(X \times X)$ defined independently of the coordinate system by $det(y^k_i) \neq 0$ with *source projection* $\alpha_q : \Pi_q \to X : (x, y_q) \to (x)$ and *target projection* $\beta_q : \Pi_q \to X : (x, y_q) \to (y)$. We shall sometimes introduce a copy Y of X with local coordinates (y) in order to avoid any confusion between the source and the target manifolds. Let us start with a Lie pseudogroup $\Gamma \subset aut(X)$ defined by a system $\mathcal{R}_q \subset \Pi_q$ of order q. In all the sequel we shall suppose that the system is involutive (see next section) and that Γ is *transitive* that is $\forall x, y \in X, \exists f \in \Gamma, y = f(x)$ or, equivalently, the map $(\alpha_q, \beta_q) : \mathcal{R}_q \to X \times X : (x, y_q) \to (x, y)$ is surjective.
2. The Lie algebra $\Theta \subset T$ of infinitesimal transformations is then obtained by linearization, setting $y = x + t\xi(x) + ...$ and passing to the limit $t \to 0$ in order to obtain the linear involutive system $R_q = id_q^{-1}(V(\mathcal{R}_q)) \subset J_q(T)$ by reciprocal image with $\Theta = \{\xi \in T | j_q(\xi) \in R_q\}$.
3. Passing from source to target, we may *prolong* the vertical infinitesimal transformations $\eta = \eta^k(y)\frac{\partial}{\partial y^k}$ to the jet coordinates up to order q in order to obtain:

$$\eta^k(y)\frac{\partial}{\partial y^k} + \frac{\partial \eta^k}{\partial y^r} y^r_i \frac{\partial}{\partial y^k_i} + (\frac{\partial^2 \eta^k}{\partial y^r \partial y^s} y^r_i y^s_j + \frac{\partial \eta^k}{\partial y^r} y^r_{ij}) \frac{\partial}{\partial y^k_{ij}} + ...$$

   where we have replaced $j_q(f)(x)$ by y_q, each component beeing the "formal" derivative of the previous one .
4. As $[\Theta, \Theta] \subset \Theta$, we may use the Frobenius theorem in order to find a generating fundamental set of *differential invariants* $\{\Phi^\tau(y_q)\}$ up to order q which are such that $\Phi^\tau(\bar{y}_q) = \Phi^\tau(y_q)$ by using the chain rule for derivatives whenever $\bar{y} = g(y) \in \Gamma$ acting now on Y. Of course, in actual practice *one must use sections of R_q instead of solutions* but it is only in section 6 that we shall see why the use of the Spencer operator will be crucial for this purpose. Specializing the Φ^τ at $id_q(x)$ we obtain the *Lie form* $\Phi^\tau(y_q) = \omega^\tau(x)$ of $\mathcal{R}_q$.
5. The main discovery of Vessiot, fifty years in advance, has been to notice that the prolongation at order q of any horizontal vector field $\xi = \xi^i(x)\frac{\partial}{\partial x^i}$ commutes with the prolongation at order q of any vertical vector field $\eta = \eta^k(y)\frac{\partial}{\partial y^k}$, exchanging therefore the differential invariants. Keeping in mind the well known property of the Jacobian determinant while passing to the finite point of view, any (local) transformation $y = f(x)$ can be lifted to a (local) transformation of the differential invariants between themselves of the form $u \to \lambda(u, j_q(f)(x))$ allowing to introduce a *natural bundle* $\mathcal{F}$ over X by patching changes of coordinates $\bar{x} = \varphi(x), \bar{u} = \lambda(u, j_q(\varphi)(x))$. A section ω of $\mathcal{F}$ is called a *geometric object* or *structure* on X and transforms like $\bar{\omega}(f(x)) = \lambda(\omega(x), j_q(f)(x))$ or simply $\bar{\omega} = j_q(f)(\omega)$. This is a way to generalize vectors and tensors ($q = 1$) or even connections ($q = 2$). As a byproduct we have $\Gamma = \{f \in aut(X) | \Phi_\omega(j_q(f)) = j_q(f)^{-1}(\omega) = \omega\}$ as a new way to write out the Lie form and we may say that Γ *preserves* ω. We also obtain $\mathcal{R}_q = \{f_q \in \Pi_q | f_q^{-1}(\omega) = \omega\}$. Coming back to the infinitesimal point of view and setting $f_t = exp(t\xi) \in aut(X), \forall \xi \in T$, we may define the *ordinary Lie derivative* with value in