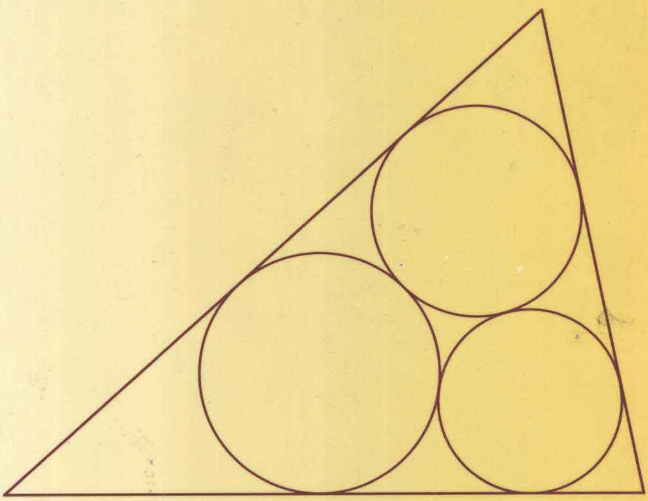
George E. Martin

GEOMETRIC CONSTRUCTIONS



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George E. Martin

Geometric Constructions

With 112 figures

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To Margaret

Books by the Author

The Foundations of Geometry
and the Non-Euclidean Plane

Transformation Geometry,
An Introduction to Symmetry

Polyominmoes,
A Guide to Puzzles and Problems in Tiling

Geometric Constructions

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Preface

> Books are to be called for and supplied on the assumption that the process of reading is not half-sleep, but in the highest sense an exercise, a gymnastic struggle; that the reader is to do something for himself.
>
> WALT WHITMAN

The old games are the best games. One of the oldest is geometric constructions. As specified by Plato, the game is played with a ruler and a compass, where the ruler can be used only to draw the line through two given points and the compass can be used only to draw the circle with a given center and through a given point. Skilled players of the game sometimes give themselves a handicap, such as restricting the compass to a fixed opening. A more severe restriction is to use only the ruler, after drawing exactly one circle (Chapter 6). On the other hand, a master player of Plato's game need not use the ruler at all (Chapter 3). Some prefer to play the game of geometric constructions with other tools, even toothpicks (Chapter 8). The most famous of the other construction tools is the marked ruler, which is simply a ruler with two marks on its edge (Chapter 9). We can do more constructions with only the marked ruler than with the ruler and compass. For example, we will prove that angle trisection is generally impossible with only the ruler and compass (Chapter 2), and we will see how to trisect any given angle with a marked ruler. The first chapter starts from scratch and reminds us of all the euclidean constructions from high school that we have forgotten or never seen. The last chapter covers geometric constructions by paperfolding.

Although many of our construction problems are inherited from antiquity, we take advantage of modern algebra and the resultant coordinate geometry to analyze and classify these problems. We necessarily encounter algebra in exploring the constructions. Various geometric construction tools are associated with various algebraic fields of numbers. This book is about these associations. Some readers will find this theoretical association a fascinating end in itself. Some will be stimulated to seek out elegant means of accomplishing those constructions that the theory proves exist and will know to avoid those proposed constructions that the theory proves do not exist. It is important to know what cannot be done in order to avoid wasting time in attempting impossible constructions. The reader of this book will not be among those few persons who turn up every year to proclaim they have "solved a construction problem that has stumped mathematicians for over two thousand years." The principal purposes for reading this book are to learn a little geometry and a little algebra and to enjoy the exercise.

Very little mathematical background is required of the reader. Abstract algebra, in general, and galois theory, in particular, are not prerequisite. Once the ideas introduced in the second chapter become familiar, the rest of the book follows smoothly. Even though the format is that of a textbook, there are so many hints and answers to be found in the lengthy section called The Back of the Book that the individual studying alone should have no problem testing comprehension against some of the exercises. A lozenge ◊ indicates that a given exercise has an entry in The Back of the Book.

By skipping over the optional Chapter 8 to get to the essential Chapter 9, an instructor can expect to cover the material in one semester. A new instructor should be warned that, although students will at first balk at the schemes that are introduced in the first chapter, the students will very quickly learn to use them and that the instructor's problem will be turning the schemes off when they are no longer appropriate.

If the figures in the text have a home-made look, it is partly because they have been made by an author learning to use *The Geometer's Sketchpad, Dynamic Geometry for the Macintosh*, published by Key Curriculum Press. The dynamic power of this software helped in making the figures and suggests a challenging follow-up seminar that attacks the question, What points can theoretically be constructed with this software? The task would be to consider the mathematical aspects of formulating a new chapter with the geometric construction tool motivated by *The Geometer's Sketchpad.*

The material has been class tested for many semesters with a master's level class for secondary teachers. The students in these classes have helped shape this book. The text jelled in the summer of 1984 with the then new Macintosh. Notes from that time show that we had class elections to determine the official definitions for the semester. The preliminary version of the text then carried the dedication FOR GILLYGALOOS EVERYWHERE. A residue of these classes can be seen in the somewhat unconventional Chapter 7, where there is a possibility of hands-on learning about mathematical structure.

I would like to thank the editors at Springer-Verlag for accepting *Geometric Constructions* for this distinguished series. There are three wonderful women at Springer-Verlag New York who have steered the text from manuscript to bound book. They are Ina Lindemann, Anne Fossella, and Victoria Evarretta. I also wish to thank Mademoiselle Claude Jacir, Documentaliste au Musée, l'Ordre de la Légion d'honneur, for providing information on Pierre Joseph Glotin. Finally, I am very much indebted to my friend and colleague Hugh Gordon, who made many helpful suggestions while teaching from preliminary versions of this book.

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Contents

1
Euclidean Constructions

> There is much to be said in favor of a game you play alone ... the company is most congenial and perfectly matched in skill and intelligence, and there is no embarrassing sarcastic utterance should you make a stupid play. The game is particularly good if it is truly challenging and if it possesses manifold variety. ... The Greek geometers of antiquity devised such a game ...
>
> HOWARD EVES

Until recently, Euclid's name and the word *geometry* were synonymous. It was Euclid who first placed mathematics on an axiomatic basis. He did such a remarkable job of presenting much of the known mathematical results of his time in such an excellent format that almost all the mathematical works that preceded his were discarded. Euclid's principal work, *Elements*, has been the dominating text in mathematics for twenty-three centuries. It is only in this last half of our own century that Euclid is not the primary text used by beginning students. Yet, we know almost nothing about this person who wrote the world's most successful secular book. We suppose Euclid studied at Plato's Academy in Athens and was an early member of the famed Museum/Library at Alexandria. Alexandria became the most important city in the Western world after the death of Alexander the Great and remained so until Caesar's Rome dominated Cleopatra's City. Even then, while Rome was at its height, Alexandria remained the intellectual capital of the Empire. Alexandria was a major influence for a thousand years, from the time of Euclid in 300 BC until the fall of Alexandria to the Arabs in AD 641. Greek mathematics is mostly a product of the Golden Age of Greek Science and Mathematics, which was centered at Alexandria in the third century BC. Although located in what is now Egypt, the ancient city was Greek with the full name Alexandria-near-Egypt. The first of the city's rulers who could even speak to the Egyptians in their own language was Cleopatra, who died in 30 BC. Can there be any doubt that the very learned Cleopatra studied her geometry from Euclid's *Elements*?

Euclid's *Elements* is divided into thirteen Books, preceded by the Axioms and the Postulates. Although there has been a great deal written about the difference between these two types of fundamental assumptions, today we no longer debate about which should be which and use the words *axiom* and *postulate* interchangeably to denote an underlying assumption.

A *definition* is an abbreviation. Definitions may abbreviate mathematical concepts with symbols as well as with words, which are, after all, also symbols. For example, assuming we understand what it means to talk about the vertices of a triangle, the mathematical symbol $\triangle ABC$ is defined to be an abbreviation for "the triangle whose vertices are the points A,B,C." Some maintain that the principal art of creating mathematics is formulating the proper definitions. All mathematics students know that the first thing that must be mastered in any mathematics class is the definitions. Otherwise, the student fails to understand what is being discussed. As Socrates said, The beginning of wisdom is the definition of terms.

Each Book of Euclid's *Elements* contains a sequence of Propositions, which are of two types: the theorems and the problems. In general, a *theorem* is a statement that has a proof based on a given set of postulates and previously proved theorems. A *proof* is a convincing argument. A *problem* in Euclid asks that some new geometric entity be created from a given set. We call a solution to such a problem a *construction*. This construction is itself a theorem, requiring a proof and having the form of a recipe: If you do this, this, and this, then you will get that. Such a mathematical recipe is called an algorithm. So a construction is the special type of theorem that is also an algorithm. (We hesitatingly offer the analogy: *Problem*: Make a pudding; *Construction*: Recipe; *Proof*: Eating.)

It may be worthwhile to reread the preceding paragraph. For a simple example of this important Problem–Construction–Proof sequence, we can take Euclid's first proposition: *Problem: Given two points A and B, construct a point C such that $\triangle ABC$ is equilateral.* After introducing the notation P_Q for the circle with center P that passes through point Q, the appropriate theorem is easily stated: *Construction: If A and B are two points and if C is one of the points of intersection of the circles A_B and B_A, then $\triangle ABC$ is equilateral.* In particular, the construction not only states the existence of the point C required by the problem but also ex-

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plicitly tells how to find C. The argument for this construction is short: *Proof: Since* $AC = AB$ *and* $BC = BA$ *because radii of the same circle are congruent, then* $AB = BC = CA$.

In addition to the Problem–Construction–Proof sequence, we should not overlook the fun of actually representing the theorem by creating an illustration that is carefully drawn with the geometric tools. Usually this drawing is also called a *construction* (The pudding?). So "construction" has two technical meanings. We will use the word for both meanings. Whether a particular occurrence of the word means the special type of theorem that is a geometric algorithm or means a drawing that illustrates such a theorem must be determined from the context. It is the combination of the construction (theorem) and the construction (drawing) that has provided so much pleasure to so many persons for so many centuries. You may feel that one is incomplete without the other.

The term *sketch* is reserved in this book for an informal, freehand representation of a formal construction drawing. Sketches are usually sufficient for our purposes. As a final observation about words and their meanings, we note that there will probably be no confusion with the general use of the word *figure* and its technical use, where the word denotes a set of points in the plane.

Whether a constructed drawing or a freehand sketch is used to illustrate a theorem, we have almost certainly been warned often that we should not argue from the figure. Some have even suggested that there should be no figures in geometry to avoid the temptation to this fault. However, figures not only help us keep track of complicated algorithms, but they are fun to see and fun to create. Geometry without figures is possible but not enjoyable. This is especially true when the geometry concerns the topic of geometric constructions.

We will use the notation in the adjacent box throughout.

A–B–C	means point B is between points A and C.
$\overleftrightarrow{AB}$	denotes the line through the two points A and B.
$\overrightarrow{AB}$	denotes the ray with vertex A that passes through B.
$\overline{AB}$	denotes the segment with endpoints A and B.
AB	denotes the distance from A to B.
A_B	denotes the circle through B with center A.
A_{BC}	denotes the circle with center A and radius BC.
$m\angle ABC$	denotes the degree measure of the angle $\angle ABC$.
$\triangle ABC$	denotes the triangle with vertices A,B,C.
ABC	denotes the area of $\triangle ABC$.
$\square ABCD$	denotes the quadrilateral with sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$.
$ABCD$	denotes the area of $\square ABCD$.
$p = q$	means "p" and "q" are names for the same object.
$p \cong q$	is read "p is congruent to q."
$p \sim q$	is read "p is similar to q."

By "given a segment," we suppose the two endpoints of a segment are determined. In the absence of further explanation, we understand "given $\overleftrightarrow{AB}$" to mean only that the two points A and B are determined.

Almost everyone knows that "Q.E.D." signifies the end of an argument. The end of the proof of a theorem in Euclid is traditionally noted by this abbreviation for "quod erat demonstratum" (which was to be proved). However, not many know what "Q.E.F." signifies. This is short for "quod erat factorum" (which was to be constructed) and comes at the end of the proof of a construction that solves a problem. Of course, this notation is a tradition from the Latin translations and not the original Greek. Today, the end of a proof is generally denoted by a symbol like the one we will use: ■.

For easy reference, we are going to state in one location many of the construction problems from Euclid. These are selected from the thirteen Books that constitute the *Elements.* A Roman numeral is used to denote the Book in which a proposition can be found. For example, "VI.8" denotes the eighth proposition in Book VI of Euclid's *Elements.*

All the Construction Problems from Book I

Euclid I.1. Construct an equilateral triangle having a given segment as one side.

Euclid I.2. Construct a segment congruent to a given segment and with a given point as one endpoint.

Euclid I.3. Given $\overrightarrow{AB}$ and $\overline{CD}$, construct point E on $\overrightarrow{AB}$ such that $\overline{AE} \cong \overline{CD}$.

Euclid I.9. Construct the angle bisector of a given angle.

Euclid I.10. Construct the midpoint of a given segment.

Euclid I.11. Given $\overleftrightarrow{AB}$, construct the perpendicular to $\overleftrightarrow{AB}$ at A.

Euclid I.12. Given point C off $\overleftrightarrow{AB}$, construct the perpendicular to $\overleftrightarrow{AB}$ that passes through C.

Euclid I.22. Construct a triangle having sides respectively congruent to three given segments whose lengths are such that the sum of the lengths of any two is greater than the length of the third.

Euclid I.23. Given $\overline{AB}$ and $\angle CDE$, construct a point F such that $\angle FAB \cong \angle CDE$.

Euclid I.31. Through a given point, construct the parallel to a given line.

Euclid I.42. Construct a parallelogram having an angle congruent to a given angle and having the area of a given triangle.

Euclid I.44. Construct a parallelogram having a given segment as a side, having an angle congruent to a given angle, and having the area of a given triangle.

Euclid I.45. Construct a parallelogram having an angle congruent to a given angle and having the area of a given polygon.

Euclid I.46. Construct a square having a given segment as one side.

Construction Problems Selected from Books II, III, and VI

Euclid II.11. Given $\overline{AB}$, construct point X on $\overline{AB}$ such that $(AB)(BX) = (AX)^2$.

Euclid II.14. Construct a square with an area equal to that of a given polygon.

Euclid III.1. Given three noncollinear points, construct the center of the circle containing the three points.

Euclid III.17. Through a given point outside a given circle, construct a tangent to the circle.

Euclid VI.12. Construct a fourth proportional to three given segments.

Euclid VI.13. Construct a mean proportional to two given segments.

All the Propositions of Book IV

Euclid IV.1. In a given circle, inscribe a chord congruent to a given segment that is shorter than a diameter.

Euclid IV.2. In a given circle, inscribe a triangle equiangular with a given triangle.

Euclid IV.3. About a given circle, circumscribe a triangle equiangular with a given triangle.

Euclid IV.4. In a given triangle, inscribe a circle.

Euclid IV.5. About a given triangle, circumscribe a circle.

Euclid IV.6. In a given circle, inscribe a square.

Euclid IV.7. About a given circle, circumscribe a square.

Euclid IV.8. In a given square, inscribe a circle.

Euclid IV.9. About a given square, circumscribe a circle.

Euclid IV.10. Construct an isosceles triangle having base angles that are double the third angle.

Euclid IV.11. In a given circle, inscribe a regular pentagon.

Euclid IV.12. About a given circle, circumscribe a regular pentagon.

Euclid IV.13. In a given regular pentagon, inscribe a circle.

Euclid IV.14. About a given regular pentagon, circumscribe a circle.

Euclid IV.15. In a given circle, inscribe a regular hexagon.

Euclid IV.16. In a given circle, inscribe a regular pentadecagon.

What tools are available for these constructions? Although we will consider other possibilities in later chapters, even the answer "the ruler and the compass" may need some explaining. A *euclidean ruler* is used only to draw the line through any two given points. A physical model of the euclidean ruler has no marks on it and is sometimes called a *straightedge*. Of course, such a physical model is necessarily of finite length, unlike the ideal euclidean ruler. For us, the word *ruler* alone will always mean a euclidean ruler. Constructions using a marked ruler will be considered in Chapter 9. A *dividers* is the drafter's tool that accomplishes the construction for Euclid I.3; a dividers is used to "carry distance." Constructions using a dividers will be investigated in Chapter 5. The *modern compass*, the compass we buy in the school supply department (and which is completely adequate for any of our needs here), serves as a dividers as well as for the purpose of drawing circles. With a modern compass, we can draw a circle having a given center and having radius the length of a given segment. The *euclidean compass*, on the other hand, can be used only to draw the circle that passes through a given point and that has a given point as its center. Note that a euclidean compass has the peculiarity of collapsing when lifted and cannot be used as a dividers. For two reasons, the word *compass* alone will always mean a modern compass for us. First, it is difficult to imagine a physical model of a euclidean compass. The second and more important reason is that Euclid's constructions for I.1, I.2, and I.3 require only a ruler and a euclidean compass. Euclid shows in his first two propositions that the same circles can be constructed with a ruler and euclidean compass as can be constructed with a ruler and modern compass, although it may take more operations using the euclidean compass. The third proposition then shows that the euclidean compass also has the power to "carry distance."

Euclid I.2. Construct a segment congruent to a given segment and with a given point as one endpoint.

Euclid I.3. Given $\overrightarrow{AB}$ and $\overline{CD}$, construct point E on $\overrightarrow{AB}$ such that $\overline{AE} \cong \overline{CD}$.

With a ruler and modern compass, I.3 has the obvious solution in the notation above of letting E be the intersection of $\overrightarrow{AB}$ and A_{CD}. However, the trick is to construct A_{CD} using a ruler and a euclidean compass. This is what I.2 is all about and a solution is not as obvious. So I.2 is exactly what is required so that I.3 follows from one more use of a euclidean compass. Once we have a ruler and euclidean compass solution for I.2, we will be able to conclude that the ruler and euclidean compass together are equivalent to the ruler and modern compass. We now restate I.2 to introduce some notation for the given points and then give Euclid's construction.

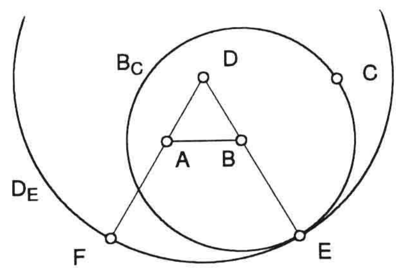
> **Problem:** Given point A and segment $\overline{BC}$, construct $\overline{AF}$ such that $\overline{AF} \cong \overline{BC}$.
> **Construction:** Given three points A,B,C, let D be a point of intersection of circles A_B and B_A. Let E be the point of intersection of B_C and $\overrightarrow{DB}$ such that B is between D and E. Let F be the point of intersection of D_E and $\overrightarrow{DA}$. Then $\overline{AF} \cong \overline{BC}$.
> **Proof:** Since $DA + AF = DF = DE = DB + BE = DA + BC$, then $AF = BC$. ∎

This proof is a bare bones version. You may want to fill out the argument with more detail. Actually, the proofs of most of the constructions in this book will be left to the reader. Usually, once a construction is stated, a proof is not too difficult to find. The hard part is to find the construction in the first place, even though the problem may have several solutions. Euclid's construction for I.2 is much clearer if you look at Figure 1.1. You may have noticed a nice convention that helps us follow the details of a construction, namely that new points are usually named in alphabetical order. With this convention in mind, you should be able to write down the construction theorem by looking at the construction drawing alone.

You see the drawing and the statement of the problem in Figure 1.1. By now you are aware that the statement of a problem is not unique; you have seen I.2 in three different forms. You should be wondering what the scheme below the statement of I.2 in Figure 1.1 is all about. This is a shorthand notation for the construction stated in words above. The scheme

$$\begin{array}{|c|} p,\ q \\ \hline P,\ Q \end{array}$$

is short for "Let P and Q be points of intersection of figures p and q." Associate each of the four sentences of the displayed section above that is labeled "Construction" with one of the four parts of the scheme in Figure 1.1; the last part of the scheme simply declares F to be what we were looking for. If there are any restrictions on the points, such as that B be between D and E, then these are written into the scheme as is also illustrated in Figure 1.1. For practice in understanding these schemes, Exercise 1.1 should be done now.



Euclid 1.2: Given three points A,B,C, construct point F such that $AF = BC$.

A_B, B_A	$B_C, \overrightarrow{DB}$	$D_E, \overrightarrow{DA}$	F
C	E D–B–E	F	

FIGURE 1.1.

Exercise 1.1. Draw sketches and write out in words the constructions for Euclid I.9 and I.10 given by the schemes below.

Euclid I.9. Construct the angle bisector of $\angle ABC$.

$B_A, \overrightarrow{BC}$	A_B, D_B	$\overrightarrow{BE}$
D	B, E	

Euclid I.10. Construct M, the midpoint of $\overline{AB}$.

A_B, B_A	$\overline{AB}, \overline{CD}$	M
C, D	M	

For practice in making the schemes, Exercise 1.2 should be done at this time.

Exercise 1.2. Give constructions in the form of a scheme for Euclid I.11 and I.12.

Euclid I.11. Given $\overleftrightarrow{AB}$, construct $\overleftrightarrow{AD}$ such that $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$.

Euclid I.12. Given point C off $\overleftrightarrow{AB}$, construct $\overleftrightarrow{CD}$ such that $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$.

Check your answers to Exercises 1.1 and 1.2 with those given in The Back of the Book. You are required to be able to read and understand these schemes. However, you are not required to use them for yourself, since you can always write out the constructions in sentences. Indeed, after the elementary constructions are assumed, it is certainly preferable to write "Let M be the midpoint of $\overline{AB}$" than to write down the scheme for the

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