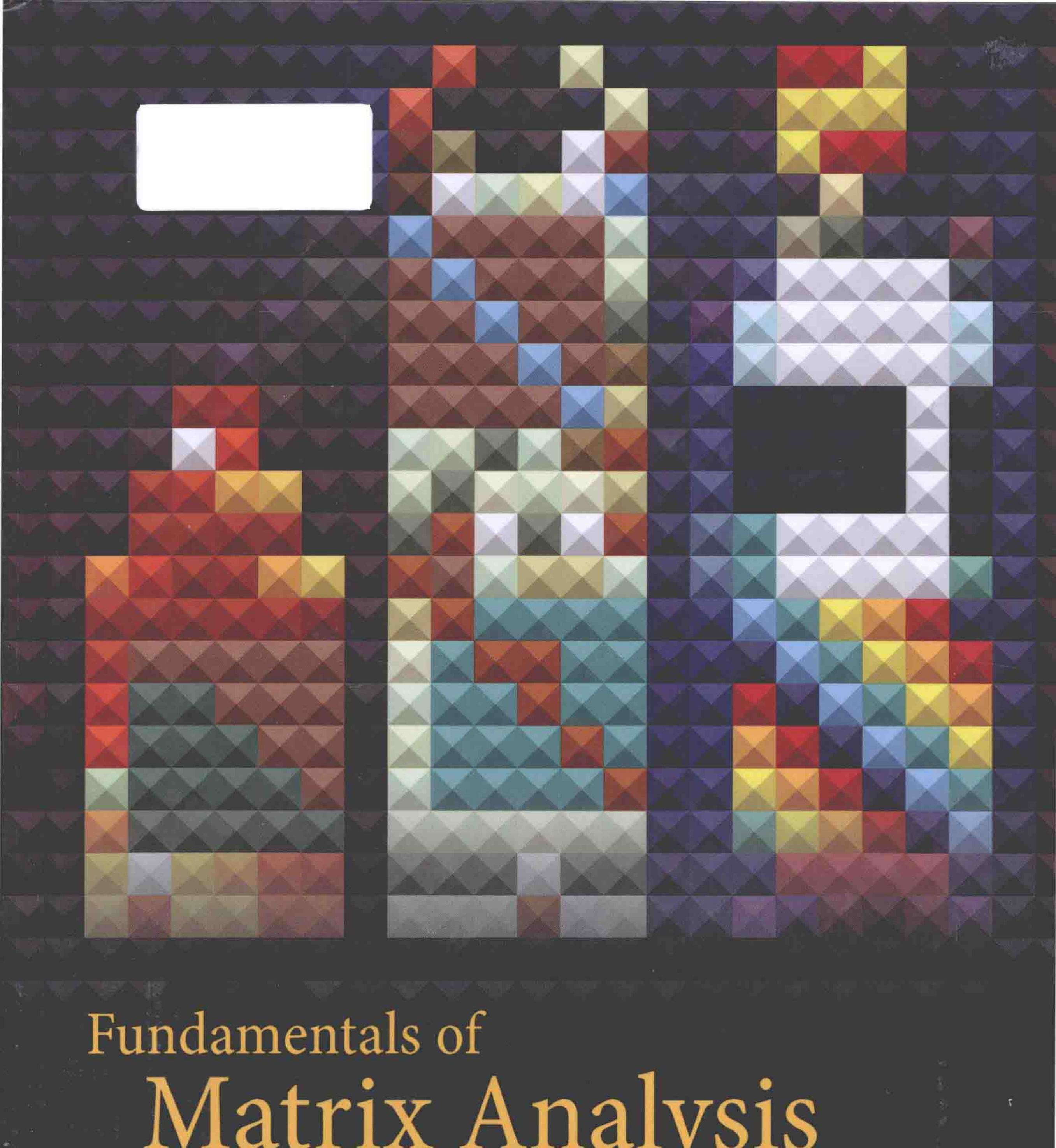

Fundamentals of Matrix Analysis with Applications

Edward Barry Saff • Arthur David Snider

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FUNDAMENTALS OF MATRIX ANALYSIS WITH APPLICATIONS

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To our brothers Harvey J. Saff, Donald J. Saff, and Arthur Herndon Snider. They have set a high bar for us, inspired us to achieve, and extended helping hands when we needed them to reach for those greater heights.

Edward Barry Saff
Arthur David Snider

PREFACE

Our goal in writing this book is to describe matrix theory from a geometric, physical point of view. The beauty of matrices is that they can express so many things in a compact, suggestive vernacular. The drudgery of matrices lies in the meticulous computation of the entries. We think matrices are beautiful.

So we try to describe each matrix operation pictorially, and squeeze as much information out of this picture as we can before we turn it over to the computer for number crunching.

Of course we want to be the computer's master, not its vassal; we want to know what the computer is doing. So we have interspersed our narrative with glimpses of the computational issues that lurk behind the symbology.

Part I. The initial hurdle that a matrix textbook author has to face is the exposition of Gauss elimination. Some readers will be seeing this for the first time, and it is of prime importance to spell out all the details of the algorithm. But students who have acquired familiarity with the basics of solving systems of equations in high school need to be stimulated occasionally to keep them awake during this tedious (in their eyes) review. In Part I, we try to pique the interests of the latter by inserting tidbits of information that would not have occurred to them, such as operation counts and computer timing, pivoting, complex coefficients, parametrized solution descriptions of underdetermined systems, and the logical pitfalls that can arise when one fails to adhere strictly to Gauss's instructions.

The introduction of matrix formulations is heralded both as a notational shorthand and as a quantifier of physical operations such as rotations, projections, reflections, and Gauss's row reductions. Inverses are studied first in this operator context before addressing them computationally. The determinant is cast in its proper light as an important concept in theory, but a cumbersome practical tool.

Readers are guided to explore projects involving LU factorizations, the matrix aspects of finite difference modeling, Kirchhoff's circuit laws, GPS systems, and fixed point methods.

Part II. We show how the vector space concepts supply an orderly organizational structure for the capabilities acquired in Part I. The many facets of orthogonality are stressed. To maintain computational perspective, a bit of attention is directed to the numerical issues of rank fragility and error control through norm preservation. Projects include rotational kinematics, Householder implementation of QR factorizations, and the infinite dimensional matrices arising in Haar wavelet formulations.

Part III. We devote a lot of print to physical visualizations of eigenvectors—for mirror reflections, rotations, row reductions, circulant matrices—before turning to the tedious issue of their calculation via the characteristic polynomial. Similarity transformations are viewed as alternative interpretations of a matrix operator; the associated theorems address its basis-free descriptors. Diagonalization is heralded as a holy grail, facilitating scads of algebraic manipulations such as inversion, root extraction, and power series evaluation. A physical experiment illustrating the stability/instability of principal axis rotations is employed to stimulate insight into quadratic forms.

Schur decomposition, though ponderous, provides a valuable instrument for understanding the orthogonal diagonalizability of normal matrices, as well as the Cayley–Hamilton theorem.

Thanks to invaluable input from our colleague Michael Lachance, Part III also provides a transparent exposition of the properties and applications of the singular value decomposition, including rank reduction and the pseudoinverse.

The practical futility of eigenvector calculation through the characteristic polynomial is outlined in a section devoted to a bird's-eye perspective of the QR algorithm. The role of luck in its implementation, as well as in the occurrence of defective matrices, is addressed.

Finally, we describe the role of matrices in the solution of linear systems of differential equations with constant coefficients, via the matrix exponential. It can be mastered before the reader has taken a course in differential equations, thanks to the analogy with the simple equation of radioactive decay. We delineate the properties of the matrix exponential and briefly survey the issues involved in its computation.

The interesting question here (in theory, at least) is the exponential of a defective matrix. Although we direct readers elsewhere for a rigorous proof of the Jordan decomposition theorem, we work out the format of the resulting exponential. Many authors ignore, mislead, or confuse their readers in the calculation of the generalized eigenvector Jordan chains of a defective matrix, and we describe a straightforward and foolproof procedure for this task. The alternative calculation of the matrix exponential, based on the primary decomposition theorem and forgoing the Jordan chains, is also presented.

Group projects for Part III address positive definite matrices, Hessenberg forms, the discrete Fourier transform, and advanced aspects of the singular value decomposition.

Each part includes summaries, review problems, and technical writing exercises.

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CONTENTS

PART I

INTRODUCTION: THREE EXAMPLES

Antarctic explorers face a problem that the rest of us wish we had. They need to consume lots of calories to keep their bodies warm. To ensure sufficient caloric intake during an upcoming 10-week expedition, a dietician wants her team to consume 2300 ounces of milk chocolate and 1100 ounces of almonds. Her outfitter can supply her with chocolate almond bars, each containing 1 ounce of milk chocolate and 0.4 ounces of almonds, for \$1.50 apiece, and he can supply bags of chocolate-covered almonds, each containing 2.75 ounces of chocolate and 2 ounces of almonds, for \$3.75 each. (For convenience, assume that she can purchase either item in fractional quantities.) How many chocolate bars and covered almonds should she buy to meet the dietary requirements? How much does it cost?

If the dietician orders x_1 chocolate bars and x_2 covered almonds, she has $1x_1 + 2.75x_2$ ounces of chocolate, and she requires 2300 ounces, so

$$1x_1 + 2.75x_2 = 2300. \tag{1}$$

Similarly, the almond requirement is

$$0.4x_1 + 2x_2 = 1100. \tag{2}$$

You're familiar with several methods of solving simultaneous equations like (1) and (2): graphing them, substituting one into another, possibly even using determinants. You can calculate the solution to be $x_1 = 1750$ bars of chocolate and $x_2 = 200$ bags of almonds at a cost of $\$1.50x_1 + \$3.75x_2 = \$3375.00$.

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But did you see that for \$238.63 *less*, she can *meet or exceed* the caloric requirements by purchasing 836.36... bags of almonds and no chocolate bars? We'll explore this in Problem 20, Exercises 1.3. Simultaneous equations, and the *linear algebra* they spawn, contain a richness that will occupy us for the entire book.

Another surprising illustration of the variety of phenomena that can occur arises in the study of differential equations.

Two differentiations of the function $\cos t$ merely result in a change of sign; in other words, $x(t) = \cos t$ solves the second-order differential equation $x'' = -x$. Another solution is $\sin t$, and it is easily verified that every combination of the form

$$x(t) = c_1 \cos t + c_2 \sin t,$$

where c_1 and c_2 are arbitrary constants, is a solution. Find values of the constants (if possible) so that $x(t)$ meets the following specifications:

$$x(0) = x(\pi/2) = 4; \tag{3}$$
$$x(0) = x(\pi) = 4; \tag{4}$$
$$x(0) = 4;\ x(\pi) = -4. \tag{5}$$

In most differential equations textbooks, it is shown that solutions to $x'' = -x$ can be visualized as vibratory motions of a mass connected to a spring, as depicted in Figure I.1. So we can interpret our task as asking if the solutions can be timed so that they pass through specified positions at specified times. This is an example of a *boundary value problem* for the differential equation. We shall show that the three specifications lead to entirely different results.

Evaluation of the trigonometric functions in the expression $x(t) = c_1 \cos t + c_2 \sin t$ reveals that for the conditions (3) we require

$$\begin{aligned} c_1 \cdot (1) + c_2 \cdot (0) &= 4 \\ c_1 \cdot (0) + c_2 \cdot (1) &= 4 \end{aligned} \tag{3'}$$

with the obvious solution $c_1 = 4$, $c_2 = 4$. The combination $x(t) = 4\cos t + 4\sin t$ meets the specifications, and in fact, it is the only such combination to do so.

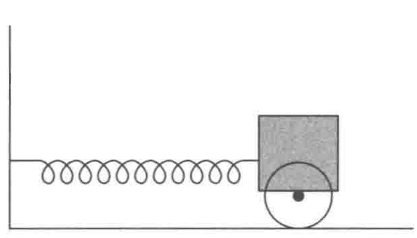

Fig. I.1 Mass-spring oscillator.

Conditions (4) require that

$$\begin{aligned} c_1 \cdot (1) + c_2 \cdot (0) &= 4, \\ c_1 \cdot (-1) + c_2 \cdot (0) &= 4, \end{aligned} \tag{4$'$}$$

demanding that c_1 be equal both to 4 and to -4. The specifications are incompatible, so no solution $x(t)$ can satisfy (4).

The requirements of system (5) are

$$\begin{aligned} c_1 \cdot (1) + c_2 \cdot (0) &= 4, \\ c_1 \cdot (-1) + c_2 \cdot (0) &= -4. \end{aligned} \tag{5$'$}$$

Both equations demand that c_1 equals 4, but no restrictions at all are placed on c_2. Thus there are infinite number of solutions of the form

$$x(t) = 4\cos t + c_2 \sin t.$$

Requirements (1, 2) and (3′, 4′, 5′) are examples of *systems of linear algebraic equations*, and although these particular cases are quite trivial to analyze, they demonstrate the varieties of solution categories that are possible. We can gain some perspective of the complexity of this topic by looking at another application governed by a linear system, namely, Computerized Axial Tomography (CAT).

The goal of a "CAT" scan is to employ radiation transmission measurements to construct a map of flesh density in a cross section of the human body. Figure I.2 shows the final resuls of a scan through a patient's midsection; experts can detect the presence of cancer tumors by noting unusual variations in the density.

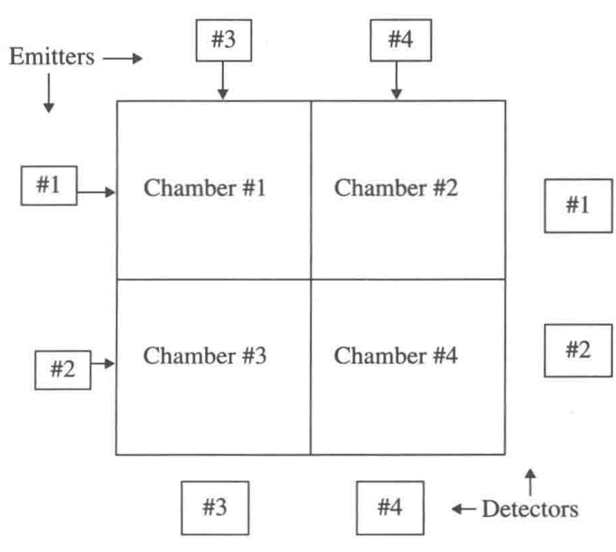

Fig. I.2 Simplified CAT scan model.

A simplified version of the technology is illustrated in Figure I.2. The stomach is modeled very crudely as an assemblage of four chambers, each with its own density. (An effective three-dimensional model for detection of tiny tumors would require millions of chambers.) A fixed dose of radiation is applied at each of the four indicated emitter locations in turn, and the amounts of radiation measured at the four detectors are recorded. We want to deduce, from this data, the flesh densities of the four subsections.

Now each chamber transmits a fraction r_i of the radiation that strikes it. Thus if a unit dose of radiation is discharged by emitter #1, a fraction r_1 of it is transmitted through chamber #1 to chamber #2, and a fraction r_2 of that is subsequently transmitted to detector #1. From biochemistry we can determine the flesh densities if we can find the transmission coefficients r_i.

So if, say, detector #1 measures a radiation intensity of 50%, and detectors #2, #3, and #4 measure intensities of 60%, 70%, and 55% respectively, then the following equations hold:

$$r_1 r_2 = 0.50; \; r_3 r_4 = 0.60; \; r_1 r_3 = 0.70; \; r_2 r_4 = 0.55.$$

By taking logarithms of both sides of these equations and setting $x_i = \ln r_i$, we find

$$\begin{aligned} x_1 + x_2 \qquad\qquad &= \ln 0.50 \\ x_3 + x_4 &= \ln 0.60 \\ x_1 \qquad + x_3 \qquad &= \ln 0.70 \\ x_2 + \qquad x_4 &= \ln 0.55, \end{aligned} \tag{6}$$

which is a system of linear algebraic equations similar to (1, 2) and (3′, 4′, 5′). But the efficient solution of (6) is much more daunting—awesome, in fact, when one considers that a realistic model comprises over 10^6 equations, and that it may possess no solutions, an infinity of solutions, or one unique solution.

In the first few chapters of this book, we will see how the basic tool of linear algebra—namely, the matrix–can be used to provide an efficient and systematic algorithm for analyzing and solving such systems. Indeed, matrices are employed in virtually every academic discipline to formulate and analyze questions of a quantitative nature. Furthermore, in Chapter Seven, we will study how linear algebra also facilitates the description of the underlying structure of the solutions of linear differential equations.