

MATHEMATICAL METHODS FOR PHYSICISTS

A concise introduction

TAIL CHOW

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Preface

This book evolved from a set of lecture notes for a course on ‘Introduction to Mathematical Physics’, that I have given at California State University, Stanislaus (CSUS) for many years. Physics majors at CSUS take introductory mathematical physics before the physics core courses, so that they may acquire the expected level of mathematical competency for the core course. It is assumed that the student has an adequate preparation in general physics and a good understanding of the mathematical manipulations of calculus. For the student who is in need of a review of calculus, however, Appendix 1 and Appendix 2 are included.

This book is not encyclopedic in character, nor does it give in a highly mathematical rigorous account. Our emphasis in the text is to provide an accessible working knowledge of some of the current important mathematical tools required in physics.

The student will find that a generous amount of detail has been given mathematical manipulations, and that ‘it-may-be-shown-thats’ have been kept to a minimum. However, to ensure that the student does not lose sight of the development underway, some of the more lengthy and tedious algebraic manipulations have been omitted when possible.

Each chapter contains a number of physics examples to illustrate the mathematical techniques just developed and to show their relevance to physics. They supplement or amplify the material in the text, and are arranged in the order in which the material is covered in the chapter. No effort has been made to trace the origins of the homework problems and examples in the book. A solution manual for instructors is available from the publishers upon adoption.

Many individuals have been very helpful in the preparation of this text. I wish to thank my colleagues in the physics department at CSUS.

Any suggestions for improvement of this text will be greatly appreciated.

Turlock, California
2000

TAI L. CHOW

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Vector and tensor analysis

Vectors and scalars

Vector methods have become standard tools for the physicists. In this chapter we discuss the properties of the vectors and vector fields that occur in classical physics. We will do so in a way, and in a notation, that leads to the formation of abstract linear vector spaces in Chapter 5.

A physical quantity that is completely specified, in appropriate units, by a single number (called its magnitude) such as volume, mass, and temperature is called a scalar. Scalar quantities are treated as ordinary real numbers. They obey all the regular rules of algebraic addition, subtraction, multiplication, division, and so on.

There are also physical quantities which require a magnitude and a direction for their complete specification. These are called vectors *if* their combination with each other is commutative (that is the order of addition may be changed without affecting the result). Thus not all quantities possessing magnitude and direction are vectors. Angular displacement, for example, may be characterised by magnitude and direction but is not a vector, for the addition of two or more angular displacements is not, in general, commutative (Fig. 1.1).

In print, we shall denote vectors by boldface letters (such as \mathbf{A}) and use ordinary italic letters (such as A) for their magnitudes; in writing, vectors are usually represented by a letter with an arrow above it such as \vec{A} . A given vector \mathbf{A} (or \vec{A}) can be written as

$$\mathbf{A} = A\hat{A}, \quad (1.1)$$

where A is the magnitude of vector \mathbf{A} and so it has unit and dimension, and \hat{A} is a dimensionless unit vector with a unity magnitude having the direction of \mathbf{A} . Thus $\hat{A} = \mathbf{A}/A$.

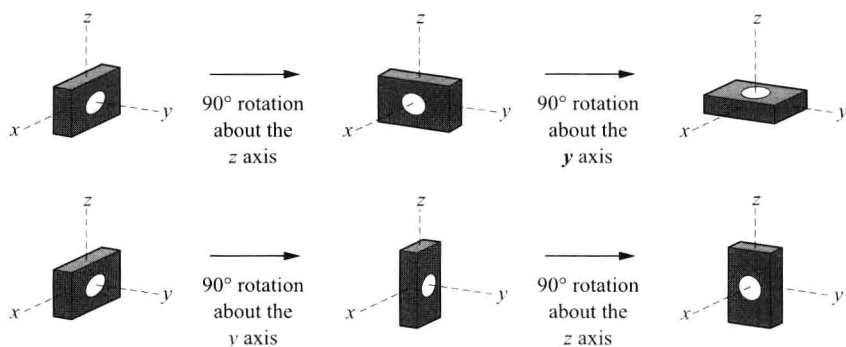


Figure 1.1. Rotation of a parallelepiped about coordinate axes.

A vector quantity may be represented graphically by an arrow-tipped line segment. The length of the arrow represents the magnitude of the vector, and the direction of the arrow is that of the vector, as shown in Fig. 1.2. Alternatively, a vector can be specified by its components (projections along the coordinate axes) and the unit vectors along the coordinate axes (Fig. 1.3):

$$\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3 = \sum_{i=1}^3 A_i\hat{e}_i, \quad (1.2)$$

where \hat{e}_i ($i = 1, 2, 3$) are unit vectors along the rectangular axes x_i ($x_1 = x$, $x_2 = y$, $x_3 = z$); they are normally written as \hat{i} , \hat{j} , \hat{k} in general physics textbooks. The component triplet (A_1, A_2, A_3) is also often used as an alternate designation for vector \mathbf{A} :

$$\mathbf{A} = (A_1, A_2, A_3). \quad (1.2a)$$

This algebraic notation of a vector can be extended (or generalized) to spaces of dimension greater than three, where an ordered n -tuple of real numbers, (A_1, A_2, \dots, A_n) , represents a vector. Even though we cannot construct physical vectors for $n > 3$, we can retain the geometrical language for these n -dimensional generalizations. Such abstract “vectors” will be the subject of Chapter 5.



Figure 1.2. Graphical representation of vector \mathbf{A} .

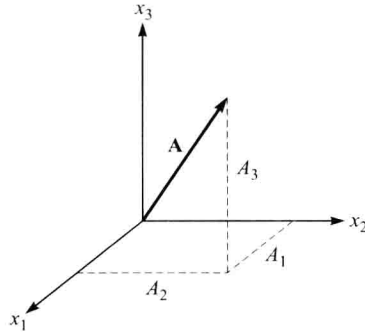


Figure 1.3. A vector \mathbf{A} in Cartesian coordinates.

Direction angles and direction cosines

We can express the unit vector \hat{A} in terms of the unit coordinate vectors \hat{e}_i . From Eq. (1.2), $\mathbf{A} = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$, we have

$$\mathbf{A} = A \left(\frac{A_1}{A} \hat{e}_1 + \frac{A_2}{A} \hat{e}_2 + \frac{A_3}{A} \hat{e}_3 \right) = A\hat{A}.$$

Now $A_1/A = \cos \alpha$, $A_2/A = \cos \beta$, and $A_3/A = \cos \gamma$ are the direction cosines of the vector \mathbf{A} , and α , β , and γ are the direction angles (Fig. 1.4). Thus we can write

$$\mathbf{A} = A(\cos \alpha \hat{e}_1 + \cos \beta \hat{e}_2 + \cos \gamma \hat{e}_3) = A\hat{A};$$

it follows that

$$\hat{A} = (\cos \alpha \hat{e}_1 + \cos \beta \hat{e}_2 + \cos \gamma \hat{e}_3) = (\cos \alpha, \cos \beta, \cos \gamma). \tag{1.3}$$

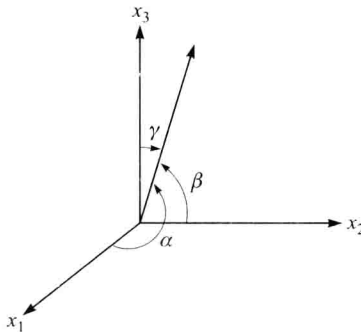


Figure 1.4. Direction angles of vector \mathbf{A} .