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Ludwig D. Faddeev . Leon A. Takhtajan

Hamiltonian Methods in the Theory of Solitons

孤立子理论中的哈密顿方法

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Translated from the Russian
by Alexey G. Reyman

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Preface

This book presents the foundations of the inverse scattering method and its applications to the theory of solitons in such a form as we understand it in Leningrad.

The concept of soliton was introduced by Kruskal and Zabusky in 1965. A soliton (a solitary wave) is a localized particle-like solution of a nonlinear equation which describes excitations of finite energy and exhibits several characteristic features: propagation does not destroy the profile of a solitary wave; the interaction of several solitary waves amounts to their elastic scattering, so that their total number and shape are preserved. Occasionally, the concept of the soliton is treated in a more general sense as a localized solution of finite energy. At present this concept is widely spread due to its universality and the abundance of applications in the analysis of various processes in nonlinear media. The inverse scattering method which is the mathematical basis of soliton theory has developed into a powerful tool of mathematical physics for studying nonlinear partial differential equations, almost as vigorous as the Fourier transform.

The book is based on the Hamiltonian interpretation of the method, hence the title. Methods of differential geometry and Hamiltonian formalism in particular are very popular in modern mathematical physics. It is precisely the general Hamiltonian formalism that presents the inverse scattering method in its most elegant form. Moreover, the Hamiltonian formalism provides a link between classical and quantum mechanics. So the book is not only an introduction to the classical soliton theory but also the groundwork for the quantum theory of solitons, to be discussed in another volume.

The book is addressed to specialists in mathematical physics. This has determined the choice of material and the level of mathematical rigour. We hope that it will also be of interest to mathematicians of other specialities and to theoretical physicists as well. Still, being a mathematical treatise it does not contain applications of soliton theory to specific physical phenomena.

While the book was written in Leningrad, the contents passed through several revisions caused by new developments of the method. We hope that in its present version the text has reached sufficient steadiness. At the same time, we do not claim to give an exhaustive account of the current state of the subject. In this sense the book is an introduction to the subject rather than an outline of all modern constructions connected with multi-dimensional generalizations and representations of infinite-dimensional algebraic structures.

We would like to thank our colleagues at the laboratory of mathematical problems of physics at the Leningrad branch of V. A. Steklov Mathematical Institute: V. E. Korepin, P. P. Kulish, A. G. Reyman, N. Yu. Reshetikhin, M. A. Semenov-Tian-Shansky, E. K. Sklyanin, F. A. Smirnov. The book undoubtedly gained from our contacts. We are also grateful to V. O. Tarasov for his careful reading of the manuscript.

Contents

Introduction

Over the past fifteen years the theory of solitons and the related theory of integrable nonlinear evolution equations in two space-time dimensions has attracted a large number of research workers of different orientations ranging from algebraic geometry to applied hydrodynamics. Modern mathematical physics has witnessed the development of a vast new area of research devoted to this theory and called the inverse scattering method of solving nonlinear equations (other names are: the inverse spectral transform, the method of isospectral deformations and, more colloquially, the L-A pair method).

The method was initiated by the pioneering work of the Princeton group. In 1967 in the paper "Method for solving the Korteweg-de Vries equation" [GGKM 1967] Gardner, Greene, Kruskal and Miura introduced a remarkable nonlinear change of variables which made the equation linear and explicitly solvable. The change of variables involves the direct and inverse scattering problems for the one-dimensional Schrödinger equation, which accounts for the name of the method.

The formation of the theory was greatly influenced by the following two contributions. In "Integrals of nonlinear equations of evolution and solitary waves" [L 1968] Lax formalized the results of the Princeton group and introduced the concept of an L-A pair. Next, in "Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media" [ZS 1971] Zakharov and Shabat showed that the concept of an L-A pair is not necessarily tied to the Korteweg-de Vries equation but can also be used for the nonlinear Schrödinger equation, thus opening perspectives for treating other equations.

Since then the increasingly fast development of the inverse scattering method and its applications has created a large new domain of mathematical physics. Characteristically, most of the work in this field is collective. Several long-standing groups can be listed besides the one in Princeton (of course, some of the people have subsequently moved to other locations). They are:

1. The group in Moscow represented by Zakharov, Manakov, Novikov, Krichever, Dubrovin and Mikhailov. Later they were joined by Gelfand, Manin and Perelomov with their collaborators.

2. The group in Potsdam represented by Ablowitz, Kaup, Newell, Segur and their collaborators.

3. The group in Arizona which includes Flaschka, Lamb and McLaughlin.

More recently a group appeared in Kyoto (Sato, Miwá, Jimbo, Kashiwara et al.). There are also other centers: in New York (Lax, Moser, Kac, McKean, Case, Deift and Trubowitz), in Rome (Calogero and Degasperis), in Manchester (Bullough with collaborators), in Freiburg (Pohlmeyer and Honerkamp). There is a group in Leningrad too, which includes the authors of the present book and also Korepin, Kulish, Reyman, Sklyanin, Semenov-Tian-Shansky, Izergin, Its and Matveev. Besides the groups some single contributors should be mentioned, Shabat, Kostant, Adler and van Moerbeke among them.

So far we have only listed mathematical physicists without mentioning the large army of specialists engaged in applications of soliton theory in quantum field theory, solid state physics, nonlinear optics, plasma physics, hydrodynamics, biology and other natural sciences. This impressive list of people and topics is indicative of the range of interests and geographical spread of those involved.

At present soliton theory is believed to have reached maturity. The increasingly prominent role of this theory was an impetus for the appearance of many monographs in which the schools mentioned above made known their particular views on the subject. They are the following:

1. Zakharov, Manakov, Novikov, Pitaievski, Theory of Solitons. The Inverse Problem Method [ZMNP 1980].

2. Lamb, Elements of Soliton Theory [L 1980].

3. Ablowitz, Segur, Solitons and the Inverse Scattering Transform [AS 1981].

4. Calogero, Degasperis, Spectral Transform and Solitons [CD 1982].

5. Dodd, Eilbeck, Gibbon, Morris, Solitons and Nonlinear waves [DEGM 1982].

There are also the following collections of papers:

1. Solitons in Action, Lonngren and Scott, eds. [LS 1978].

2. Solitons, Bullough and Caudrey, eds. [BC 1980].

3. Bäcklund Transformations, Miura, ed. [M 1976].

4. Proceedings of the Joint US-USSR symposium on Soliton Theory, Manakov and Zakharov, eds. [MZ 1981].

5. Nonlinear Evolution equations Solvable by the Spectral Transform, Calogero, ed. [C 1978].

There is also a textbook of Eilenberger "Solitons: Mathematical Methods for Physicists" [E 1981].

After publishing a number of reviews devoted to the quantum theory of solitons and its applications in quantum field theory [KF 1977], [FK 1978], [F 1980a], [F 1980b] the Leningrad specialists think it timely to voice their attitude towards the inverse scattering method as a whole. Naturally, the attitude presented is influenced by the orientation towards the quantum formulation of soliton theory. The quantum version of the inverse scattering method which has been developed since 1978 and reviewed in a series of papers [TF 1979], [KS 1980], [F 1980b], [F 1981], [F 1982a], [F 1982b], [IK 1982], [F 1983], [T 1983] forced us to look afresh at the basic tools and devices of the classical version of the method. Particularly, this concerns the language of Hamiltonian dynamics closely associated with quantum applications.

As a matter of fact, most integrable models (including all of applied importance) possess a Hamiltonian structure, that is, the equations defining them are infinite-dimensional analogues of Hamilton's equations in classical mechanics. The inverse scattering transform can be interpreted as a canonical transformation with respect to this structure so that the variables which linearize the equation have the meaning of action-angle variables.

For the example of the Korteweg-de Vries equation this programme was formulated and carried out in the paper of Zakharov and Faddeev "Korteweg-de Vries equation, a completely integrable Hamiltonian system" [ZF 1971] published in 1971, which was the formative period of the theory. Later the same was done for other interesting models.

The treatises cited above often mention the Hamiltonian approach but never assign to it a principal methodological role. The main point in which our book differs from others is the emphasis on the Hamiltonian structure and the ensuing choice and arrangement of the material (see the Preface). At the same time the text is self-contained enough to serve as an independent introduction to the subject.

At first we planned to devote the book mainly to the quantum version, with a suitable introduction to the classical method. However, as often happens, the project expanded in the course of writing and the book will appear in two volumes. The present volume is devoted entirely to the classical theory.

The pedagogical novelties of the book are clearly noticeable. In contrast to other authors, we have chosen the nonlinear Schrödinger (NS) equation

$$i\frac{\partial \psi}{\partial t} = -\frac{\partial^2 \psi}{\partial x^2} + 2\varkappa|\psi|^2\psi,$$

where $\psi(x,t)$ is a complex-valued function, to be our principal representative example, instead of the Korteweg-de Vires (KdV) equation

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}.$$

For this there are several reasons:

1. In many technical respects the NS equation is simpler and more fundamental than the KdV equation. Thus, the NS equation illustrates directly some simple general constructions of the method, whereas their extension to the KdV equation requires a reduction procedure. In particular, the auxiliary linear problem for the NS equation (the eigenvalue problem for the Lax operator L) is a system of first order differential equations of general type. For the KdV equation, the role of the operator L is played by the one-dimensional Schrödinger operator whose spectral theory is slightly more complicated. Moreover, this operator may be regarded as a very special case of a first order system.

2. The Hamiltonian formalism for the NS equation is more simple and straightforward. The field variables $\psi(x)$ and $\bar{\psi}(x)$, the bar indicating complex conjugation, form a simple set of canonical variables,

$$\{\psi(x), \bar{\psi}(y)\} = i\delta(x-y).$$

At the same time, the Poisson brackets for the KdV equation

$$\{u(x), u(y)\} = \frac{1}{2}\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial x}\right)\delta(x-y)$$

do not immediately lead to an obvious choice of canonically conjugate variables.

3. The NS equation has a natural quantum analogue describing a quantum system of an indefinite number of particles interacting pairwise via the potentials $v_{ij} = \delta(x_i - x_j)$. Therefore it suits our project, including quantum theory, particularly well. At the same time, the KdV equation has no direct physical meaning in the quantum domain.

4. The last but not the least motivation comes from our spirit of contradiction which forbids us to begin yet another textbook with the hackneyed KdV equation.

The discussion of the NS equation occupies nearly half of the book and is organized in a separate part. We exploit this equation to present the foundations of the method in a form which would make its extension to other equations more or less automatic. All arguments are presented in detail and proofs are mathematically as rigorous as is compatible with our sense of what is reasonable. As a consequence, when analyzing other models we can simply refer to the NS equation. Only the characteristic differences of these models are discussed at some length.

Part Two is devoted to the analysis of several representative models which have played a significant role in the development of the inverse scattering method. We call them fundamental models. These are the models defined by the following equations:

1) the Sine-Gordon equation

$$\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + \frac{m^2}{\beta} \sin\beta\varphi = 0$$

for a real-valued function $\varphi(x, t)$;

2) the Heisenberg ferromagnet equation

$$\frac{\partial \vec{S}}{\partial t} = \vec{S} \wedge \frac{\partial^2 \vec{S}}{\partial x^2},$$

where $\vec{S}(x, t)$ is on the unit sphere in $\mathbb{R}^3$ and $\wedge$ denotes exterior product;

3) the Toda lattice equation

$$\frac{d^2 q_n}{dt^2} = e^{q_{n+1} - q_n} - e^{q_n - q_{n-1}}$$

for the coordinates q_n, $-\infty < q_n < \infty$.

These are the models discussed most thoroughly in the body of the book. In addition, one will encounter here several other models of physical interest (the N-wave model, the chiral field, and the Landau-Lifshitz model). Finally, in Part Two we outline a fairly general classification scheme of integrable models and methods for solving them.

From the technical point of view, the main distinctive features of our exposition are as follows:

1. Instead of the original Lax representation

$$\frac{dL}{dt} = [L, A]$$

and the corresponding auxiliary linear problem

$$L\Psi = \lambda\Psi$$

we use from the very beginning the zero curvature representation

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0$$

and the auxiliary linear problem of the form

$$\frac{\partial F}{\partial x} = U(x, \lambda) F.$$

2. Alongside the usual analysis of the direct and inverse scattering problems for the auxiliary linear system on an infinite interval we also consider the finite interval $-L \leqslant x \leqslant L$ with quasi-periodic boundary conditions. However, the associated inverse problem involves analysis on Riemann surfaces and goes beyond the scope of our book.

3. Our treatment of the inverse problem is based on the matrix Riemann problem of analytic factorization of matrix-valued functions, rather then on the traditional Gelfand-Levitan-Marchenko equation. As has now become clear, this method is more universal and technically more transparent. For the prototype NS equation we explain how the Gelfand-Levitan-Marchenko method can be naturally incorporated into the Riemann problem.

4. The Hamiltonian structure is defined in terms of the so-called r-matrix. This construction originated in the quantum spectral transform method and later was adapted to the classical case. We believe it to be most adequate and universal and hope to demonstrate this.

5. A comprehensive classification of integrable models based on the concept of an r-matrix is presented. The Lie-bracket formalism for (infinite-dimensional) current algebras turns out to provide an adequate language for continuous models. We also discuss an extension of the classification to lattice models.

We emphasize again that the above characteristic features have their natural counterparts in the quantum version of the method.

Now, a word about the level of mathematical rigour. Our presentation, mostly elementary, is based on techniques of classical analysis. Proofs are given of all results on direct and inverse problems for the auxiliary linear system for the NS model in the rapidly decreasing case. This is not done when other models are discussed in order to avoid overloading the text with tiresome details. We believe that the NS model is treated in a sufficiently invariant manner, so that the reader will be able to fill in the gaps.

However, a rigorous proof of the assertions concerning the Hamiltonian formalism should make use of analysis on infinite-dimensional manifolds. We consider this level of rigour superfluous for our subject and therefore do not hesitate to use differential-geometric terminology in the infinite-dimensional case without complete justification. This is done deliberately because in our view rigorous proofs in this situation do not reveal the heart of the matter; so we leave the job to specialists in global analysis. We believe that this agrees with the state of affairs in modern mathematical physics to which the present text belongs.

The inverse scattering method is now developed to such an extent that it can be presented from the very beginning in its most general form. However, this does not seem to be the best way of introducing the subject. As an alternative we have chosen to introduce its basic concepts by means of a particular example and to illustrate its generality by other models, so that the reader is led gradually to the fairly natural and general construction