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Recent Progress on Some Problems in Several Complex Variables and Partial Differential Equations

International Conference
Partial Differential Equations and Several Complex Variables
Wuhan University, Wuhan, China
June 9–13, 2004

International Conference
Complex Geometry and Related Fields
East China Normal University, Shanghai, China
June 21–24, 2004

Shiferaw Berhanu, Hua Chen,
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Preface

This volume consists mainly of papers presented at two related conferences held in Wuhan University and the East China Normal University at Shanghai in the Summer of 2004. Several participants from many institutions throughout the world attended these international conferences. The conference in Wuhan focused on recent developments in many areas of several complex variables and partial differential equations. The conference in Shanghai celebrated the 10th anniversary of the Institute of Mathematics at East China Normal University and the focus was complex geometry and related topics. Many researchers attended both conferences and as a result we decided to publish the proceedings in one volume.

We are very grateful to Wuhan University, the NNSF of China, the State Education Ministry of China and the East China Normal University at Shanghai for their generous support to host these conferences.

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Problems on the Monge-Ampère Equation in the Plane

Claudia Anedda and Giovanni Porru

Abstract. We prove that the classical Makar-Limanov function corresponding to a convex solution of the Monge-Ampère equation in dimension 2 satisfies an interior maximum principle and an interior minimum principle. Since such a function is a constant in case the domain is an ellipse, we have a best possible maximum principle. As application we discuss a special overdetermined boundary value problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with a smooth boundary $\partial\Omega$, and let u be the negative (and convex) solution of the Dirichlet problem

$$u_{11}u_{22} - u_{12}u_{21} = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.1}$$

where u_i denotes partial differentiation with respect to x^i, $i = 1, 2$. If $H = [u_{ij}]$ is the 2×2 Hessian matrix associated with u, the Newton matrix T is defined as

$$T = \Delta u I - H, \tag{1.2}$$

where $\Delta u = u_{11} + u_{22}$ and I is the unit matrix. If $T = [T^{ij}]$ one finds

$$T^{11} = u_{22}, \qquad T^{12} = T^{21} = -u_{12}, \qquad T^{22} = u_{11}. \tag{1.3}$$

We prove that the function

$$M = T^{ij}u_iu_j - 2u \tag{1.4}$$

assumes its maximum value on the boundary $\partial\Omega$ and its minimum value either on the boundary or at a critical point of u. Here and in the sequel the summation convention over repeated indexes is used. Observe that when Ω is an ellipse then M is a constant in Ω. Therefore, our result is a best possible maximum principle in the sense of [**3**]. We apply this maximum principle to discuss the following overdetermined boundary value problem: find domains Ω such that the solution u to problem (1.1) satisfies the additional condition

$$T^{ij}u_iu_j = c^2 \ \ (c = constant) \quad \text{on } \partial\Omega. \tag{1.5}$$

To motivate this unusual boundary condition we describe an optimization problem in which it arises. We prove that if a convex solution to problem (1.1) exists and satisfies condition (1.5) then Ω must be an ellipse. To get this result we follow a

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method introduced by H. Weinberger in [**11**] to discuss the following overdetermined boundary value problem:

$$u_{11} + u_{22} = 1 \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad |\nabla u| = c \text{ on } \partial\Omega. \tag{1.6}$$

In [**11**] it is proved that if a solution to problem (1.6) exists then Ω must be a disc. The same result has been obtained by J. Serrin in [**9**] by using the moving plane method.

Using the Monge-Ampère equation (1.1), the function M defined in (1.4) can be rewritten as

$$M = T^{ij}\big(u_i u_j - u\, u_{ij}\big). \tag{1.7}$$

This function has been introduced by Makar-Limanov in [**2**], where it is proved that if u is a solution to the equation $u_{11}+u_{22} = 1$ then the corresponding function (1.7) attains its maximum value on the boundary of Ω. Unfortunately, this interesting result holds (for the Laplace equation) in dimension two only. At the end of this paper we propose an extension of the Makar-Limanov function defined for solutions of a special fully non linear equation.

We emphasize that the function M involves the second derivatives of the solution u. Other functions involving the second derivatives which satisfy a best possible maximum principle are described in [**5**], [**6**]. For functions depending on u and ∇u which satisfy a best possible maximum principle we refer to [**3**] and [**4**] and references therein.

2. Main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain and let u be a smooth solution of the Monge-Ampère equation

$$u_{11}u_{22} - u_{12}u_{21} = 1 \quad \text{in } \Omega. \tag{2.1}$$

Let H be the 2×2 Hessian matrix corresponding to u. Since the Newton matrix T defined as in (1.2) is the cofactor matrix of H, we have $T = \det(H)H^{-1}$, where H^{-1} is the inverse matrix of H. Therefore the equation (2.1) implies

$$T = H^{-1}. \tag{2.2}$$

It follows that

$$T^{ij}u_{ij} = \text{trace}(H^{-1}H) = 2.$$

Hence, the classical Makar-Limanov function (1.7), for solutions to (2.1) can be rewritten as

$$M = T^{ij}u_i u_j - 2u. \tag{2.3}$$

Lemma 2.1. *If u satisfies the Monge-Ampère equation* (2.1) *then we have*

$$T^{hk}M_{hk} = T^{hk}(T^{ij})_{hk}u_i u_j.$$

Proof. We find

$$M_h = (T^{ij})_h u_i u_j + 2T^{ij}u_{ih}u_j - 2u_h.$$

Since by (2.2) $TH = I$, we have

$$2T^{ij}u_{ih}u_j = 2\delta_h^j u_j = 2u_h,$$

where δ_h^j is the usual Kronecker delta. Therefore,

$$M_h = (T^{ij})_h u_i u_j. \tag{2.4}$$

If we derive with respect to x^k we find

$$M_{hk} = (T^{ij})_{hk} u_i u_j + 2(T^{ij})_h u_{ik} u_j. \tag{2.5}$$

Now, using (2.2) and since

$$(T^{ij})_i = (\Delta u)_i \; \delta^i_j - u_{iji} = (\Delta u)_j - (\Delta u)_j = 0,$$

we find

$$T^{hk}(T^{ij})_h u_{ik} = \delta^h_i (T^{ij})_h = (T^{ij})_i = 0.$$

The lemma follows by using (2.5) and the previous equations. □

THEOREM 2.2. *If u is a convex solution to the Monge-Ampère equation* (2.1) *in a domain $\Omega \subset \mathbb{R}^2$ then the function M defined in* (2.3) *assumes its maximum value on $\partial\Omega$, and its minimum value either on $\partial\Omega$ or at the point where u attains its minimum.*

PROOF. Recalling that

$$T^{11} = u_{22}, \qquad T^{12} = T^{21} = -u_{12}, \qquad T^{22} = u_{11},$$

Lemma 2.1 yields

$$\begin{aligned} T^{hk} M_{hk} &= T^{hk} u_{22hk} \, (u_1)^2 - 2T^{hk} u_{12hk} \, u_1 u_2 + T^{hk} u_{11hk} \, (u_2)^2 \\ &= (u_{22}u_{1122} - 2u_{12}u_{1222} + u_{11}u_{2222})(u_1)^2 \\ &\quad - 2(u_{22}u_{1112} - 2u_{12}u_{1122} + u_{11}u_{1222})u_1 u_2 \\ &\quad + (u_{22}u_{1111} - 2u_{12}u_{1112} + u_{11}u_{1122})(u_2)^2. \end{aligned}$$

By equation (2.1) we find

$$u_{111}u_{22} + u_{122}u_{11} - 2u_{112}u_{12} = 0, \tag{2.6}$$

and

$$u_{112}u_{22} + u_{222}u_{11} - 2u_{122}u_{12} = 0. \tag{2.7}$$

Further differentiation yields

$$\begin{gathered} u_{1111}u_{22} + 2u_{111}u_{122} + u_{11}u_{1122} - 2u_{112}u_{112} - 2u_{12}u_{1112} = 0, \\ u_{1112}u_{22} + u_{111}u_{222} - u_{112}u_{122} + u_{11}u_{1222} - 2u_{12}u_{1122} = 0, \\ u_{1122}u_{22} + 2u_{112}u_{222} + u_{11}u_{2222} - 2u_{122}u_{122} - 2u_{12}u_{1222} = 0. \end{gathered}$$

Using the last three equations, we find

$$\begin{aligned} T^{hk} M_{hk} &= 2(u_{122}u_{122} - u_{112}u_{222})(u_1)^2 \\ &\quad - 2(u_{112}u_{122} - u_{111}u_{222})u_1 u_2 + 2(u_{112}u_{112} - u_{111}u_{122})(u_2)^2. \end{aligned} \tag{2.8}$$

By (2.4) we have

$$\begin{aligned} M_1 &= u_{221}(u_1)^2 - 2u_{112}u_1u_2 + u_{111}(u_2)^2, \\ M_2 &= u_{222}(u_1)^2 - 2u_{122}u_1u_2 + u_{112}(u_2)^2. \end{aligned}$$

These two equations together with (2.6) and (2.7) give a system of four linear equations with four unknowns u_{111}, u_{112}, u_{122} and u_{222}. By computation one finds that the determinant of the corresponding coefficients equals $\left(T^{ij}u_iu_j\right)^2$. Therefore, solving such a system and using (2.8) we find

$$T^{hk} M_{hk} = B^{hk} M_h M_k,$$

where B^{hk} are regular functions for $T^{ij}u_iu_j \neq 0$. Since the solution u is assumed to be convex, the matrix $[T^{hk}]$ is positive definite. By the classical maximum principle [**7**] we infer that $M(x)$ attains its maximum and its minimum values either on $\partial\Omega$ or at the point in Ω with $\nabla u = 0$. Since $u(x)$ is convex, if we have $\nabla u = 0$ at $x_0 \in \Omega$ then x_0 is the point of minimum of $u(x)$.

We prove now that $M(x)$ must assume its maximum value on $\partial\Omega$. Arguing by contradiction, let $x_0 \in \Omega$ such that

$$M(x_0) > \max_{x\in\partial\Omega} M(x).$$

With $\epsilon > 0$ small we may suppose that $\tilde{M}(x) = M(x) + \epsilon(x^1 - x_0^1)^2$ satisfies

$$\tilde{M}(x_0) > \max_{x\in\partial\Omega} \tilde{M}(x).$$

Then, $\tilde{M}(x)$ attains its maximum value at some interior point $\tilde{x}$ and

$$T^{hk}\tilde{M}_{hk} \leq 0 \qquad \text{at } \tilde{x}. \tag{2.9}$$

On the other hand, one finds easily that

$$T^{hk}\tilde{M}_{hk} = T^{hk}M_{hk} + 2\epsilon u_{22} > T^{hk}M_{hk}. \tag{2.10}$$

In the last step we have used the inequality $u_{22} > 0$, true because $u(x)$ is convex. Let us perform a suitable rotation

$$x - \tilde{x} = C(y - \tilde{x}) : \mathbb{R}^2 \to \mathbb{R}^2$$

so that the mixed derivative u_{12} with respect to the new variables (y^1, y^2) vanishes at $\tilde{x}$. We have

$$CC^t = I, \quad y - \tilde{x} = C^t(x - \tilde{x}), \quad \nabla u = CDu,$$

where Du denotes the gradient of u with respect to the new variables (y^1, y^2). Similarly, if $\nabla^2 u$ (resp. D^2u) denotes the Hessian matrix of u with respect to (x^1, x^2) (resp. (y^1, y^2)) and if $T(\nabla^2 u)$ (resp. $T(D^2u)$) denotes the Newton tensor corresponding to $\nabla^2 u$ (resp. D^2u) we find

$$\nabla^2 u = CD^2uC^t, \qquad T(\nabla^2 u) = CT(D^2u)C^t.$$

As a consequence,

$$\begin{aligned} M(x) &= (\nabla u)^t T(\nabla^2 u)\nabla u - 2u \\ &= (Du)^t C^t T(\nabla^2 u) CDu - 2u \\ &= (Du)^t T(D^2u)Du - 2u = M(y). \end{aligned}$$

Furthermore,

$$\begin{aligned} T^{hk}(\nabla^2 u)M_{hk}(x) &= tr\{T(\nabla^2 u)\nabla^2 M\} \\ = tr\{CT(D^2u)C^t\nabla^2 M\} &= tr\{T(D^2u)C^t\nabla^2 MC\} \\ = tr\{T(D^2u)D^2M\} &= T^{hk}(D^2u)M_{hk}(y). \end{aligned}$$

It follows that equations (2.1), (2.6), (2.7) and (2.8) remain the same after the change of variables. In terms of the new variables we have $u_{12} = 0$ and $u_{11}u_{22} = 1$ at $\tilde{x}$. Then, (2.6) and (2.7) yield

$$u_{111}u_{22} + u_{122}u_{11} = 0, \qquad u_{112}u_{22} + u_{222}u_{11} = 0. \tag{2.11}$$

Equations (2.11) imply that

$$u_{112}u_{122} - u_{111}u_{222} = 0.$$

Therefore, the coefficient of u_1u_2 in (2.8) vanishes. Moreover, by the first equation in (2.11) we find

$$2u_{111}u_{122} + (u_{111}u_{22})^2 + (u_{122}u_{11})^2 = 0.$$

Similarly, by the second equation in (2.11) we find

$$2u_{112}u_{222} + (u_{112}u_{22})^2 + (u_{222}u_{11})^2 = 0.$$

Using the last equations, by (2.8) we get

$$\begin{aligned} T^{hk}M_{hk} &\\ = [2(u_{122})^2 &+ (u_{112}u_{22})^2 + (u_{222}u_{11})^2](u_1)^2 \\ +[2(u_{112})^2 &+ (u_{111}u_{22})^2 + (u_{122}u_{11})^2](u_2)^2 \geq 0. \end{aligned} \tag{2.12}$$

Hence, (2.10) implies that

$$T^{hk}\tilde{M}_{hk} > 0 \qquad \text{at } \tilde{x}.$$

The last inequality contradicts (2.9) and the proof is completed. □

LEMMA 2.3. *If Ω is convex and smooth, if ν denotes the exterior unit normal to $\partial\Omega$ and if u is a convex solution of the Dirichlet problem*

$$u_{11}u_{22} - u_{12}u_{21} = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

then we have

$$\int_{\partial\Omega} x^\ell \nu^\ell T^{ij} u_i u_j \, ds = 8 \int_\Omega (-u)\, dx.$$

PROOF. Since $u(x) = 0$ on $\partial\Omega$, if $\nu = (\nu^1, \nu^2)$ we have $\nu^j = \frac{u_j}{|\nabla u|}$ on $\partial\Omega$. Using this fact and the Green formula we find

$$\begin{aligned} \int_{\partial\Omega} x^\ell \nu^\ell T^{ij} u_i u_j \, ds &= \int_{\partial\Omega} x^\ell u_\ell T^{ij} u_i \nu^j \, ds = \int_\Omega \left(x^\ell u_\ell T^{ij} u_i\right)_j dx \\ = \int_\Omega T^{ij} u_i u_j \, dx &+ \int_\Omega x^\ell u_{\ell j} T^{ij} u_i \, dx + \int_\Omega x^\ell u_\ell \left(T^{ij}\right)_j u_i \, dx + \int_\Omega x^\ell u_\ell T^{ij} u_{ij} \, dx. \end{aligned}$$

Since $T = H^{-1}$ we have $u_{\ell j}T^{ij} = \delta^i_\ell$. We also recall that $\left(T^{ij}\right)_j = 0$. Using the Monge-Ampère equation we find

$$T^{ij}u_{ij} = 2(u_{11}u_{22} - u_{12}u_{21}) = 2.$$

Hence,

$$\int_{\partial\Omega} x^\ell \nu^\ell T^{ij} u_i u_j \, ds = \int_\Omega T^{ij} u_i u_j \, dx + 3\int_\Omega x^\ell u_\ell \, dx.$$

Since

$$\int_\Omega T^{ij} u_i u_j \, dx = -\int_\Omega T^{ij} u_{ij} u \, dx = -2\int_\Omega u \, dx$$

and

$$\int_\Omega x^\ell u_\ell \, dx = -\int_\Omega (x^\ell)_\ell u \, dx = -2\int_\Omega u \, dx,$$

the lemma follows. □

THEOREM 2.4. *If there exists a convex solution u to the Dirichlet problem*

$$u_{11}u_{22} - u_{12}u_{21} = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \tag{2.13}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^2$, and if it satisfies the additional condition

$$T^{ij}u_i u_j = c^2 \qquad \text{on} \quad \partial\Omega \tag{2.14}$$

then Ω must be an ellipse. Here c is a positive constant.

PROOF. By Lemma 2.3 we have

$$\int_{\partial\Omega} x^\ell \nu^\ell T^{ij} u_i u_j \, ds = 8 \int_\Omega (-u)\, dx.$$

Using (2.14) we find

$$\int_{\partial\Omega} x^\ell \nu^\ell T^{ij} u_i u_j \, ds = c^2 \int_{\partial\Omega} x^\ell \nu^\ell \, ds = c^2 \int_\Omega \left(x^\ell\right)_\ell dx = c^2 2A(\Omega)$$

where $A(\Omega)$ denotes the Lebesgue measure of Ω. Hence,

$$c^2 A(\Omega) = 4 \int_\Omega (-u)\, dx. \tag{2.15}$$

On the other side, using the maximum principle for M proved in Theorem 2.2 and our boundary conditions we find

$$\int_\Omega T^{ij} u_i u_j \, dx + 2\int_\Omega (-u)\, dx \le c^2 A(\Omega). \tag{2.16}$$

Integration by parts and use of (2.13) yield

$$\int_\Omega T^{ij} u_i u_j \, dx = -\int_\Omega T^{ij} u_{ij} u \, dx = 2\int_\Omega (-u)\, dx. \tag{2.17}$$

Hence, (2.16) can be rewritten as

$$4\int_\Omega (-u)\, dx \le c^2 A(\Omega).$$

Comparing the last equation with (2.15) we infer that equality must hold in (2.16). Recalling Theorem 2.2 we deduce that $M(x)$ must be a constant in Ω. But then the quadratic form (2.8) must vanish. At a point where $u_{12} = 0$, (2.8) becomes (2.12). If $u_1 \neq 0$ we find

$$u_{122} = u_{112} = u_{222} = 0.$$

By the first of (2.11) we also find $u_{111} = 0$. If $u_2 \neq 0$, by (2.12) and the second of (2.11) we again find that all third derivatives of u vanish. If $u_{12} \neq 0$ we can perform a suitable rotation in order to have $u_{12} = 0$ in terms of the new variables. One proves that all third derivatives with respect to the new variables vanish. But then, also the third derivatives with respect to the old variables must vanish provided that $|\nabla u| > 0$. Because u is assumed to be convex, the gradient vanishes at one point only (the point of minimum). At that point the third derivatives must vanish for continuity reasons. Therefore we have

$$u_{111} = u_{112} = u_{122} = u_{222} = 0 \quad \text{in} \quad \Omega.$$

By

$$u_{111} = u_{112} = 0$$

we find $u_{11} = a$. This equation together with $u_{122} = 0$ yield

$$u_1 = ax^1 + bx^2 + c,$$

where a, b and c are constants and x^1, x^2 are the coordinates of a point in Ω. Finally, since $u_{222} = 0$ we find

$$u = a_1(x^1)^2 + a_2x^1x^2 + a_3(x^2)^2 + b_1x^1 + b_2x^2 + c.$$

Since $u = 0$ on $\partial\Omega$ and since Ω is bounded, it follows that Ω must be an ellipse. □

REMARK 2.5. Let us make a comment on the boundary condition (2.14). Following [**10**], let $V = V(x) : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^2 vector field, let t be a real number and let $\Omega^t = (Id + tV)(\Omega)$, where Id denotes the identity map. If Ω is strictly convex and $|t|$ is small, also Ω^t is convex. If $u^t = u_{\Omega^t}$ is the convex solution to problem (2.13) with Ω replaced by Ω^t and if

$$J(\Omega^t) = 2\int_{\Omega^t} (-u^t)\,dx, \quad J(\Omega) = 2\int_{\Omega} (-u)\,dx,$$

the domain derivative of $J(\Omega)$ in the direction of V is defined as

$$dJ(\Omega; V) = \lim_{t\to 0} \frac{J(\Omega^t) - J(\Omega)}{t}.$$

Furthermore, the derivative $u'(x)$ in the direction of V is defined by

$$u'(x) = \lim_{t\to 0} \frac{u^t(x) - u(x)}{t}.$$

In [**10**] (pag. 680) the following boundary value of $u'(x)$ is found:

$$u'(x) = -\frac{\partial u}{\partial \nu}(V \cdot \nu) \quad \text{on } \partial\Omega. \tag{2.18}$$

As usual, ν denotes the exterior unit normal to $\partial\Omega$. Since $u = 0$ on $\partial\Omega$, the derivative of $J(\Omega)$ in the direction V is given by [**10**]

$$dJ(\Omega; V) = 2\int_{\Omega} (-u')\,dx. \tag{2.19}$$

By equation (2.13) we find

$$u'_{11}u_{22} + u_{11}u'_{22} - u'_{12}u_{21} - u_{12}u'_{21} = 0.$$

Multiplying by u and integrating over Ω, the last equation yields

$$\int_{\Omega} T^{ij}u_iu'_j\,dx = 0. \tag{2.20}$$

By equation (2.13) we also have

$$-u'T^{ij}u_{ij} = -2u'.$$

Integrating over Ω we find

$$\int_{\partial\Omega} (-u')T^{ij}u_i\nu^j\,ds + \int_{\Omega} T^{ij}u_iu'_j\,dx = 2\int_{\Omega} (-u')\,dx.$$

Since $\frac{\partial u}{\partial \nu}\nu^j = u_j$ on $\partial\Omega$, using (2.18) and (2.20) we find

$$\int_{\partial\Omega} T^{ij}u_iu_j(V \cdot \nu)\,ds = 2\int_{\Omega} (-u')\,dx.$$

Insertion of the last result into (2.19) leads to

$$dJ(\Omega; V) = \int_{\partial\Omega} T^{ij}u_iu_j(V \cdot \nu)\,ds. \tag{2.21}$$

On the other hand, if $A(\Omega)$ denotes the Lebesgue measure of Ω we have

$$dA(\Omega; V) = \int_{\partial\Omega} (V \cdot \nu)\, ds.$$

Hence, by (2.21) we find

$$dJ(\Omega; V) = c^2 dA(\Omega; V)$$

for all displacement field V if and only if condition (2.14) holds. Therefore, condition (2.14) is related with the critical points of the functional $\Omega \to J(\Omega)$ under the condition $A(\Omega) = constant$.

3. An open problem

If $u(x)$ is a smooth function defined in a domain $\Omega \subset \mathbb{R}^N$, if H is the Hessian matrix $[u_{ij}]$ and if k is an integer with $1 \le k \le N$, denote with $S_{(k)}(u)$ the k-th elementary symmetric function of the eigenvalues of H. The problem

$$S_{(k)}(u) = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{3.1}$$

has been discussed, for example, in [**1**]. For $k = 1$ we have the well known equation $\Delta u = 1$. For $1 < k \le N$, problem (3.1) has a (negative) smooth solution provided Ω satisfies a suitable convexity condition [**1**]. A natural extension of the boundary condition (1.5) for problem (3.1) is

$$T^{ij}_{(k-1)} u_i u_j = c^2 \quad \text{on} \quad \partial\Omega, \tag{3.2}$$

where

$$T^{ij}_{(k-1)} = \frac{\partial S_{(k)}}{\partial u_{ij}}.$$

Note that $T^{ij}_{(0)} = \delta^{ij}$, the familiar Kronecker delta. Therefore, for $k = 1$, condition (3.2) reduces to $|\nabla u| = c$, used in [**9**] and in [**11**]. For $1 \le k < N$, the Newton tensor $T_{(k)} = [T^{ij}_{(k)}]$ is well known in differential geometry [**8**] and satisfies

$$T^{ij}_{(k)} = \frac{1}{k!} \binom{i_1 \cdots i_k i}{j_1 \cdots j_k j} u_{i_1 j_1} \cdots u_{i_k j_k}, \tag{3.3}$$

where the generalized Kronecker symbol $\binom{i_1 \cdots i_k i}{j_1 \cdots j_k j}$ has the value 1 (respectively -1) if the indexes $i_1, \cdots, i_k, i$ are distinct and $(j_1, \cdots, j_k, j)$ is an even (respectively odd) permutation of $(i_1, \cdots, i_k, i)$, otherwise it has value zero. Recall that the summation convention over repeated indexes from 1 to N is understood.

The boundary condition (3.2) arises investigating the critical points of the functional

$$J(\Omega) = k \int_\Omega (-u)\, dx$$

for small deformations of Ω under the condition that the measure of Ω is a constant [**10**].

To investigate the overdetermined problem (3.1)-(3.2), it would be useful an interior maximum principle for the function

$$M(x) = \frac{1}{2} T^{ij}_{(k-1)} u_i u_j - \frac{k}{N} u, \tag{3.4}$$

where $u = u(x)$ is a solution to (3.1). A maximum principle for $M(x)$ when $k = 1$ is well known (see, for example, [**3**] and [**4**]), and for $N = k = 2$ it is proved in this paper.

Let us show that $M(x)$ is a constant when Ω is a ball. In this situation, the solution to problem (3.1) is radial and, if $r = |x|$ and if R is the radius of the ball, we have

$$u(r) = \frac{1}{2}\binom{N}{k}^{-\frac{1}{k}}(r^2 - R^2), \quad u' = \binom{N}{k}^{-\frac{1}{k}} r, \quad u'' = \frac{u'}{r}.$$

Moreover H is now a scalar matrix $H = \frac{u'}{r} I$. Therefore, using (3.3) we find

$$\frac{1}{2} T^{ij}_{(k-1)} u_i u_j = \frac{1}{2} \frac{1}{(k-1)!} \binom{i_1 \cdots i_{k-1} i}{i_1 \cdots i_{k-1} i} \left(\frac{u'}{r}\right)^{k-1} \left(\frac{x^i}{r}\right)^2 (u')^2$$

$$= \frac{1}{2}\binom{N-1}{k-1} (u')^{k+1} r^{1-k}.$$

Finally,

$$M(x) = \frac{1}{2}\binom{N-1}{k-1}(u')^{k+1} r^{1-k} - \frac{k}{N} u = \frac{1}{2}\frac{k}{N}\binom{N}{k}^{-\frac{1}{k}} R^2.$$

We have found that $M(x)$ is a constant in the ball.

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GLOBAL SOLVABILITY FOR A SPECIAL CLASS OF VECTOR FIELDS ON THE TORUS

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Abstract. We study the global solvability of a class of complex vector fields on the two-torus. For $\mathsf{L} = \partial/\partial t + (a(x) + ib(x))\partial/\partial x$, $a, b \in C^\infty(\mathbb{T}^1; \mathbb{R})$, we show that a necessary condition for L to be strongly solvable is that each zero of $a + ib$ is of finite order. We say that L is strongly solvable if the image of operator $\mathsf{L} : C^\infty(\mathbb{T}^2) \to C^\infty(\mathbb{T}^2)$ is closed and has finite codimension. One of the main points of our work is to shed light on the interplay between the orders of vanishing of a and b at each common zero, which is crucial for strong solvability of L.

1. Introduction

Let K be a compact subset of a smooth manifold X. As in Hörmander [H2], we say that a differential operator $P(x, D)$ in X is *solvable at K* if the equation $P(x, D)u = f$ is satisfied near K for some distribution $u \in \mathcal{D}'(X)$ for every f belonging to a finite codimensional subspace of $C^\infty(X)$.

In the present work, we deal with a related concept, namely, the operator P is said to be *strongly solvable* in $C^\infty(X)$ if the range of $P : C^\infty(X) \to C^\infty(X)$ is closed and has finite codimension. We will also refer to the following weaker notion: the operator P is said to be *globally solvable* in $C^\infty(X)$ if the range of $P : C^\infty(X) \to C^\infty(X)$ is closed.

We will study a class of complex vector fields on the two-torus $\mathbb{T}^2$, of the special form

$$\mathsf{L} = \partial/\partial t + (a(x) + ib(x))\partial/\partial x, \quad a, b \in C^\infty(\mathbb{T}^1; \mathbb{R}), \tag{1.1}$$

and we will present necessary conditions and sufficient conditions for the strong solvability in $C^\infty(\mathbb{T}^2)$ of the vector field L.

We proceed to describe some of the known results, and we begin with the case $b \equiv 0$. In [BP] the subject of study was the global solvability of $\mathsf{L} = \partial/\partial t + a(x)\partial/\partial x$; on the other hand, the strong solvability was not considered. In this case, by using some of the arguments that will appear later on in the present work, it is possible

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