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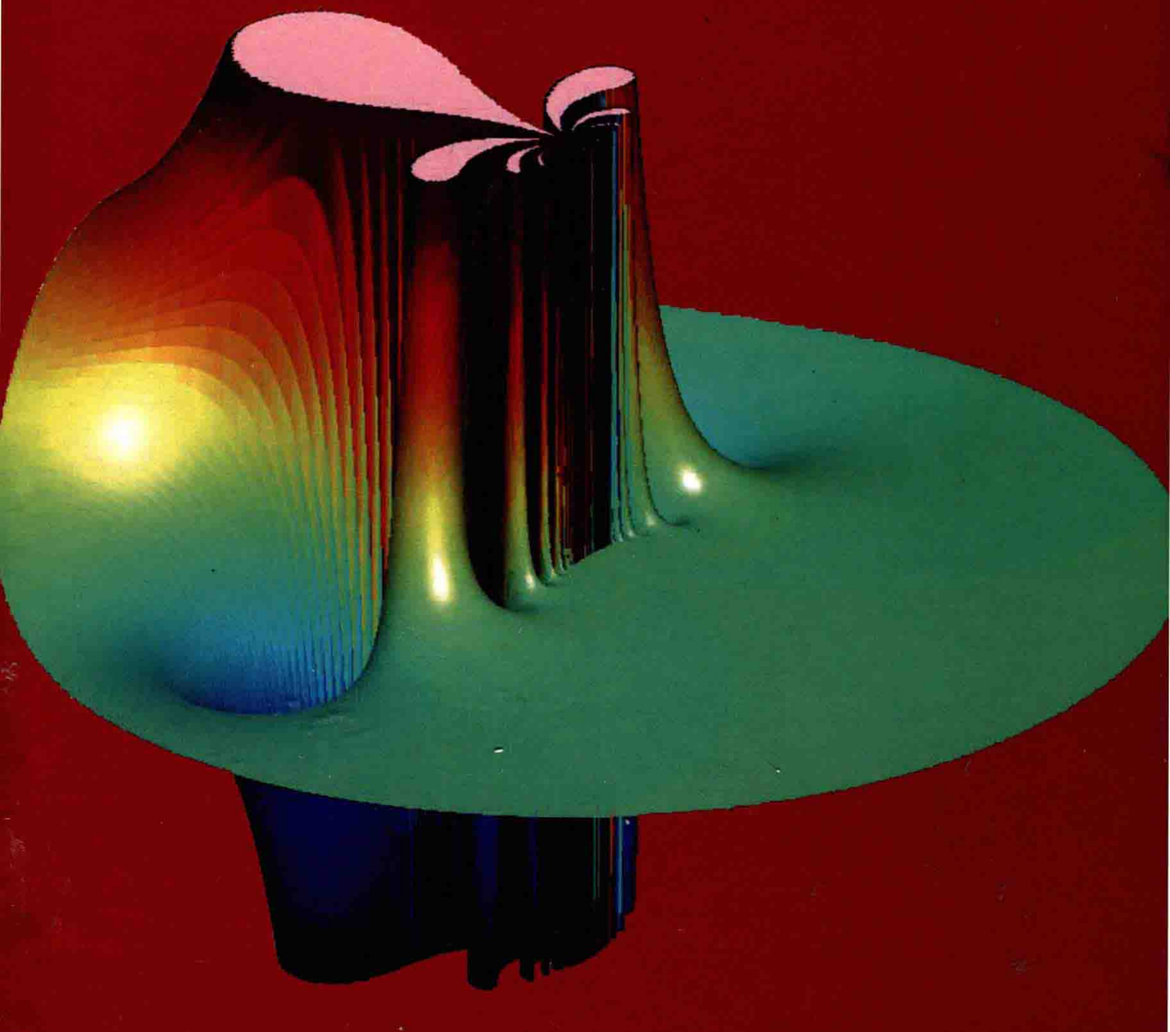
Practical Differential Equations

Sabita Mahanta



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Concepts in Practical Differential Equations

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CONCEPTS IN PRACTICAL DIFFERENTIAL EQUATIONS

Preface

The study of differential equations is a wide field in pure and applied mathematics, physics, meteorology, and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods. Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere. This extension is often necessary for solutions to exist, and it also results in more physically reasonable properties of solutions, such as possible presence of shocks for equations of hyperbolic type. The study of the stability of solutions of differential equations is known as stability theory.

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modelled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time value varies. Newton's laws allow one (given the position, velocity, acceleration and various forces acting on the body) to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly. An example of modelling a real world problem using differential equations is the determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is considered constant, and air resistance may be modelled as proportional to the ball's velocity. This means that the ball's acceleration, which is a derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation. Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions —the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

The texts are arranged in a lucid form and written in colloquial English. All the essential aspects of this subject have been included. Hopefully, the present study will prove very useful for students and teachers.

—Editor

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1

Introduction

Differential Equation

A differential equation is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines. Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modelled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and velocity as the time value varies. Newton's laws allow one (given the position, velocity, acceleration and various forces acting on the body) to express these variables dynamically as a differential equation for the unknown position of the body as a function of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

An example of modelling a real world problem using differential equations is the determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is considered constant, and air resistance may be modelled as proportional to the ball's velocity. This means that the ball's acceleration, which is a derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation.

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions — the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Directions of Study

The study of differential equations is a wide field in pure and applied mathematics, physics, meteorology, and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods.

Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere. This extension is often necessary for solutions to exist, and it also results in more physically reasonable properties of solutions, such as possible presence of shocks for equations of hyperbolic type.

The study of the stability of solutions of differential equations is known as stability theory.

Nomenclature

The theory of differential equations is well developed and the methods used to study them vary significantly with the type of the equation.

Ordinary and Partial

- An ordinary differential equation (ODE) is a differential equation in which the unknown function (also known as the dependent variable) is a function of a *single* independent variable. In the simplest form, the unknown function is a real or complex valued function, but more generally, it may be vector-valued or matrix-valued: this corresponds to considering a system of ordinary differential equations for a single function.

Ordinary differential equations are further classified according to the order of the highest derivative of the dependent variable with respect to the independent variable appearing in the equation. The most important cases for applications are first-order and second-order differential equations. For example, Bessel's differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2)y = 0$$

(in which y is the dependent variable) is a second-order differential equation. In the classical literature also distinction is made between differential equations explicitly solved with respect to the highest derivative and differential equations in an implicit form.

A partial differential equation (PDE) is a differential equation in which the unknown function is a function of *multiple* independent variables and the equation involves its partial derivatives. The order is defined similarly to the case of ordinary differential equations, but further classification into elliptic, hyperbolic, and parabolic equations, especially for second-order linear equations, is of utmost importance. Some partial differential equations do not fall into any of these categories over the whole domain of the independent variables and they are said to be of mixed type.

Linear and Non-linear

Both ordinary and partial differential equations are broadly classified as linear and nonlinear.

- A differential equation is linear if the unknown function and its derivatives appear to the power 1 (products are not allowed) and nonlinear otherwise. The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations. Homogeneous linear differential equations are a further subclass for which the space of solutions is a linear subspace i.e. the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables; if these coefficients are constants then one speaks of a constant coefficient linear differential equation.

- There are very few methods of solving nonlinear differential equations exactly; those that are known typically depend on the equation having particular symmetries. Nonlinear differential equations can exhibit very complicated behaviour over extended time intervals, characteristic of chaos. Even the fundamental questions of existence, uniqueness, and extendability of solutions for nonlinear differential equations, and well-posedness of initial and boundary value problems for nonlinear PDEs are hard problems and their resolution in special cases is considered to be a significant advance in the mathematical theory (cf. Navier–Stokes existence and smoothness).

Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations.

Examples: In the first group of examples, let u be an unknown function of x, and c and ω are known constants.

- Inhomogeneous first-order linear constant coefficient ordinary differential equation: $\frac{du}{dx} = cu + x^2.$
- Homogeneous second-order linear ordinary differential equation: $\frac{d^2u}{dx^2} - x\frac{du}{dx} + u = 0.$
- Homogeneous second-order linear constant coefficient ordinary differential equation describing the harmonic oscillator: $\frac{d^2u}{dx^2} + \omega^2 u = 0.$

- Inhomogeneous first-order nonlinear ordinary differential equation: $\frac{du}{dx} = u^2 + 1$.
- Second-order nonlinear ordinary differential equation describing the motion of a pendulum of length L: $L\frac{d^2u}{dx^2} + g\sin u = 0$.

In the next group of examples, the unknown function u depends on two variables x and t or x and y.

- Homogeneous first-order linear partial differential equation: $\frac{\partial u}{\partial t} + t\frac{\partial u}{\partial x} = 0$.
- Homogeneous second-order linear constant coefficient partial differential equation of elliptic type, the Laplace equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
- Third-order nonlinear partial differential equation, the Korteweg–de Vries equation: $\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}$.

Related Concepts

- A delay differential equation (DDE) is an equation for a function of a single variable, usually called time, in which the derivative of the function at a certain time is given in terms of the values of the function at earlier times.
- A stochastic differential equation (SDE) is an equation in which the unknown quantity is a stochastic process and the equation involves some known stochastic processes, for example, the Wiener process in the case of diffusion equations.
- A differential algebraic equation (DAE) is a differential equation comprising differential and algebraic terms, given in implicit form.

Connection to difference equations

The theory of differential equations is closely related to the theory of difference equations, in which the coordinates assume only discrete values, and the relationship involves values of the unknown function or functions and values at nearby coordinates.

Many methods to compute numerical solutions of differential equations or study the properties of differential equations involve approximation of the solution of a differential equation by the solution of a corresponding difference equation.

Universality of Mathematical Description

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics, differential equations are used to model the behaviour of complex systems. The mathematical theory of differential equations first developed together with the sciences where the equations had originated and where the results found application.

However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena.

As an example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond. All of them may be described by the same second-order partial differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat, the theory of which was developed by Joseph Fourier, is governed by another second-order partial differential equation, the heat equation. It turned out that many diffusion processes, while seemingly different, are described by the same equation; Black–Scholes equation in finance is for instance, related to the heat equation.

Operations

Differential Operator

In mathematics, a differential operator is an operator defined as a function of the differentiation operator. It is helpful, as a matter of notation first, to consider differentiation as an abstract operation, accepting a function and returning another (in the style of a higher-order function in computer science).

This article considers mainly linear operators, which are the most common type. However, non-linear differential operators, such as the Schwarzian derivative also exist.

Notations

The most common differential operator is the action of taking the derivative itself. Common notations for taking the first derivative with respect to a variable x include:

$$\frac{d}{dx}, D, D_x, \text{and} \partial_x..$$

When taking higher, nth order derivatives, the operator may also be written:

$$\frac{d^n}{dx^n}, D^n, \text{or} D_x^n.$$

The derivative of a function f of an argument x is sometimes given as either of the following:

$$[f(x)]' \; f'(x)$$

The D notation's use and creation is credited to Oliver Heaviside, who considered differential operators of the form

$$\sum_{k=0}^{n} c_k D^k$$

in his study of differential equations.

One of the most frequently seen differential operators is the Laplacian operator, defined by

$$\Delta = \nabla^2 = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

Another differential operator is the È operator, or theta operator, defined by

$$\Theta = z\frac{d}{dz}.$$

This is sometimes also called the homogeneity operator, because its eigenfunctions are the monomials in z:

$$\Theta(z^k) = kz^k, \quad k = 0, 1, 2, \ldots$$

In n variables the homogeneity operator is given by

$$\Theta = \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k}.$$

As in one variable, the eigenspaces of Θ are the spaces of homogeneous polynomials.

The result of applying the differential to the left and to the right, and the difference obtained when applying the differential operator to the left and to the right, are denoted by arrows as follows:

$$f\overleftarrow{\partial}_x g = g\partial_x f$$
$$f\overrightarrow{\partial}_x g = f\partial_x g$$
$$f\overleftrightarrow{\partial}_x g = f\partial_x g - g\partial_x f$$

Such a bidirectional-arrow notation is frequently used for describing the probability current of quantum mechanics.

Del

The differential operator del is an important vector differential operator. It appears frequently in physics in places like the differential form of Maxwell's Equations. In three dimensional Cartesian coordinates, del is defined:

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

Del is used to calculate the gradient, curl, divergence, and laplacian of various objects.

Adjoint of an Operator

Given a linear differential operator T

$$Tu = \sum_{k=0}^{n} a_k(x)D^k u$$

the adjoint of this operator is defined as the operator T^* such that

$$\langle Tu, v\rangle = \langle u, T^* v\rangle$$

where the notation $\langle\cdot,\cdot\rangle$ is used for the scalar product or inner product. This definition therefore depends on the definition of the scalar product.

Formal Adjoint in One Variable

In the functional space of square integrable functions, the scalar product is defined by

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx.$$